02.08.01 MATH 2280 Chapter 3 Linear equations of higher order

Recall from § 3.1 – 3.3:

$$L(y) = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y$$

General solution to $L(y) = f$ can be written as

$$y = y_H + y_P \rightarrow \text{§ } 3.5 \quad Ly_P = f$$

$Ly_H = 0$

When $a_i(x) = a_i$ = constants:
look at characteristic polynomial.

$$p(r) = r^n + a_{n-1}r^{n-1} + \dots + a_1 r + a_0$$

(you get this by simply plugin $y = e^{rx}$ in $L(y)=0$).

(related to notion of characteristic polynomials for matrices)

If characteristic polynomial has n distinct roots: a basis for $\ker(L)$ is:

$$\{e^{r_1 x}, e^{r_2 x}, \dots, e^{r_n x}\}$$

works even if roots are complex (beautiful Euler formula:

$$e^{(a+ib)x} = e^{ax}e^{ibx} = e^{ax}(\cos bx + i\sin bx)$$

Case of multiple roots:

If $p(r) = (r - r_1)^{k_1} q(r)$ (here degree $q = n - k_1$)

Then $\underbrace{\{e^{r_1 x}, xe^{r_1 x}, \dots, x^{k-1}e^{r_1 x}\}}_{\text{as many as multiplicity of roots}}$ are k lin indep sol to $L(y) = 0$.

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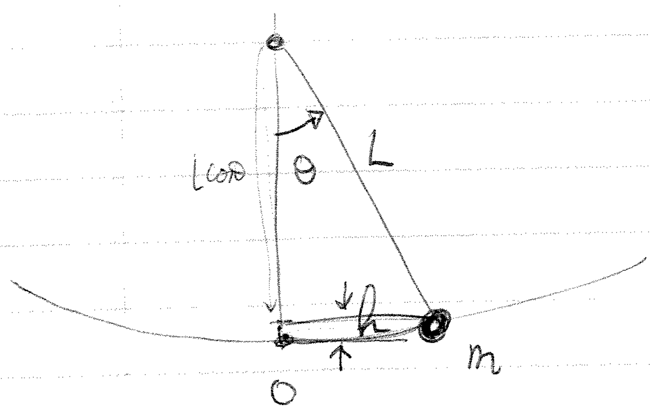
§3.4 Mechanical vibrations

We will make sense of theory with examples from physics:

- pendulum
- undamped motion
- damped motion
  - → overdamped
  - → critically damped
  - → underdamped

• <u>Pendulum</u> (simple)



$s(t) = L\theta(t)$

We will use conservation of energy:

$$E_k = \frac{1}{2} m v^2 = \frac{1}{2} m \left(\frac{ds}{dt}\right)^2$$
$$= \frac{L^2}{2} m \left(\frac{d\theta}{dt}\right)^2$$

$$E_p = mgh = mgL(1-\cos\theta)$$

$$\Rightarrow \quad E_k + E_p = C \qquad\qquad (\cos\theta)' = -\theta' \sin\theta$$

$$\frac{d}{dt} \Rightarrow \quad mL^2 \frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + mgL\frac{d\theta}{dt}\sin\theta = 0$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

Small angle approx $\sin\theta \simeq \theta$ (Taylor!)

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0.$$

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Or more generally:

$$\theta'' + c\theta' + k\theta = 0$$

$c\theta'$ accounts for friction.

Free Undamped Motion

$c = 0$ no damping

$$mx'' + kx = 0$$
$$x'' + \frac{k}{m}x = 0.$$

Characteristic polyn:

$$r^2 + \frac{k}{m} = 0$$
$$r = \pm i\sqrt{\frac{k}{m}} \quad \omega_0$$

solutions $x = A'e^{i\omega_0 t} + B'e^{-i\omega_0 t}$

$= A\cos\omega_0 t + B\sin\omega_0 t$

Let $C = \sqrt{A^2 + B^2}$

$$\Rightarrow x(t) = C\left(\underbrace{\frac{A}{C}}_{\cos\alpha}\cos\omega_0 t + \underbrace{\frac{B}{C}}_{\sin\alpha}\sin\omega_0 t\right)$$

$$= C\cos(\omega_0 t - \alpha)$$

Amplitude (C), ~~freq~~ angular freq (rad/s) (ω_0), phase angle (α)

Spring, mass and damping

~~x(0) = equi~~

$x = 0$ = equilibrium

Inventory of forces:

- Hooke's law

$F_S = -kx$ opposes displacement (compression or extension)

k: spring const

- Resistance force:

$F_R = -cv = -c\frac{dx}{dt}$

c: damping constant

opposes movement & prop to velocity.

- External force

$F_E = F(t)$

Newton's Second Law:

$$m\frac{d^2x}{dt^2} = F_S + F_R + F_E$$

$$\Rightarrow \boxed{mx'' + cx' + kx = F(t)}$$

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This is called "simple harmonic motion".

Period = time for one full oscillation

$$= T = \frac{2\pi}{\omega_0} \quad (s)$$

$$\text{frequency} = 1/\text{Period} = \frac{\omega_0}{2\pi} \quad (s^{-1})$$

Free Damped Motion

rewriting $mx'' + cx' + kx = 0$ as:

$$x'' + 2px' + \omega_0^2 x = 0$$

Characteristic polynomial:

$$r^2 + 2pr + \omega_0^2 = 0$$

roots: $$r = \frac{-2p \pm \sqrt{4p^2 - 4\omega_0^2}}{2}$$

$$= -p \pm \sqrt{p^2 - \omega_0^2}$$

Depending on sign of $p^2 - \omega_0^2$ we have several physical situations:

- $p^2 - \omega_0^2 > 0$: roots are real : overdamped case (decay exp)

$$x(t) = Ae^{r_1 t} + Be^{r_2 t}$$ Both roots are negative

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- $\underline{p^2 = \omega_0^2}$: one single real root of multiplicity 2.

$$x(t) = e^{-pt}(A + Bt)$$

dubbed: critically damped case

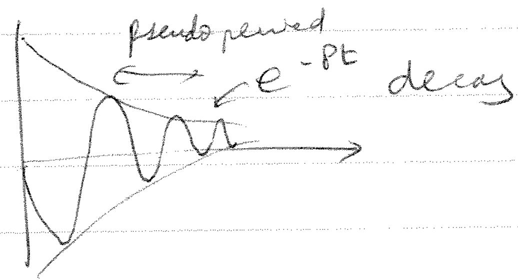
- $\underline{p^2 - \omega_0^2 < 0}$. 2 imaginary roots: underdamped case

$$r_1 = -p + i\sqrt{\omega_0^2 - p^2}$$
$$r_2 = -p - i\sqrt{\omega_0^2 - p^2}$$

$$\begin{aligned} x(t) &= Ae^{r_1 t} + Be^{r_2 t} \\ &= Ae^{-pt}e^{i\sqrt{\omega_0^2-p^2}\,t} + Be^{-pt}e^{-i\overbrace{\sqrt{\omega_0^2-p^2}}^{\omega_1}\,t} \\ &= e^{-pt}\left(Ae^{i\cancel{\sqrt{\omega_0^2-p^2}}\,\omega_1 t} + Be^{-i\omega_1 t}\right) \\ &= e^{-pt}\left(A'\cos\omega_1 t + B'\sin\omega_1 t\right) \\ &= Ce^{-pt}\cos(\omega_1 t - \alpha) \quad \text{(using same trick)} \end{aligned}$$

ω_1 : pseudo frequency

pseudo periodic motion damping exponentially.



$$T = \frac{2\pi}{\omega_1}$$

Do Example 2 in Maple.

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§ 3.5 Non homogeneous Equations and undetermined coefficients

$$L = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0$$

we know how to solve $Ly_H = 0$.

how to solve $Ly_p = f$? (So full sol: $y = y_p + y_H$)

Method of undetermined coeff (~ guessing)

Example:

$$y' + 4y = 2e^{3x}$$

$$y_H = \cancel{Ae^{2x} + Be^{2x}} \quad Ae^{-4x}$$

$\times 4$ try $y_p = Ce^{3x}$ since $L(e^{rx}) = p(r)e^{rx}$

$\times 1$ $y_p' = 3Ce^{3x}$

$\cancel{(y_p'' = 9Ce^{3x})}$

$$(3+4)\,\cancel{(\ )}\,Ce^{3x} = 2e^{3x} \Rightarrow C = \frac{2}{7}$$

$$\Rightarrow \boxed{y_p = \frac{2}{7}e^{3x}} \text{ and } \boxed{y = y_H + y_p = \cancel{Ae^{2x} + Be^{2x} + \tfrac{2}{5}e^{3x}}}$$

$$Ae^{-4x} + \frac{2}{7}e^{3x}.$$

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Example 2: $y' + 4y = \cos 2x$ $\qquad y^H_{(x)} = Ce^{-4x}$

$$V = \mathrm{span}\{\cos 2x, \sin 2x\}$$

If we look at $L: V \to V$ $\quad \ker L = \{0\}$

$\Rightarrow \exists!\ v \in V$ s.t. $Lv = \cos 2x$.

Book:

$4\times \quad y_p = A\cos 2x + B\sin 2x$

$1\times \quad y_p' = -2A\sin 2x + 2B\cos 2x$

$\Rightarrow L(y_p) = (4A + 2B)\cos 2x + (-2A + 4B)\sin 2x$

$= 1\cos 2x + 0 \cdot \sin 2x$

$L(y_p) =$

$$\begin{pmatrix} 4 & 2 \\ -2 & 4 \end{pmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 4 & 2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 1/10 \end{bmatrix}$$

$$\boxed{y_p = \frac{1}{5}\cos 2x + \frac{1}{10}\sin 2x}$$

Math 2270 way

Basis $\mathcal{B} = \{\cos 2x, \sin 2x\}$ of V. In this basis:

$$[L]_{\mathcal{B}} = \left[(L(f_1))_{\mathcal{B}}, (L(f_2))_{\mathcal{B}}\right] = \begin{bmatrix} 4 & 2 \\ -2 & 4 \end{bmatrix}$$

$L(\cos 2x) = 4\cos 2x - 2\sin 2x$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example 3 find y_p for $L(y_p) = x+2$

look for $y_p \in$ span $\{1, x\}$ $\qquad \begin{bmatrix}2\\1\end{bmatrix}$.

$$L(1) = 4 \qquad y_p = A + Bx$$

$$L(x) = 1 + 4x$$

$$(L)_{\mathcal{B}} = \begin{pmatrix}4 & 1\\0 & 4\end{pmatrix}$$

$$\begin{pmatrix}4 & 1\\0 & 4\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}A\\B\end{pmatrix} = \frac{1}{16}\begin{pmatrix}4 & -1\\0 & 4\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \frac{1}{16}\begin{pmatrix}7\\4\end{pmatrix}$$

$$\Rightarrow \boxed{y_p = \frac{7}{16} + \frac{4}{16}x.}$$

Example 4: Use above examples to solve:

$$y' + 4y = e^{3x} - 10\cos 2x + 4x + 8$$

using superposition principle: if $L(y_{p_1}) = 2e^{3x}$ then $L(\frac{1}{2}y_{p_1}) = e^{3x}$ etc....
(linear op)

$$\Rightarrow y_p = \frac{1}{7}e^{3x} - 2\cos 2x - \sin 2x + \frac{7}{4} + x$$

To summarize:

$f(x)$	guess for y_p
e^{rx}	Ce^{rx}
$e^{ax}(A\cos bx + B\sin bx)$	$e^{ax}(D\cos bx + E\sin bx)$
$P_n(x)$	$Q_n(x)$
$P_n(x)e^{rx}$	$Q_n(x)e^{rx}$
$P_n(x)(A\cos bx + B\sin bx)$	$Q_n(x)\cos bx + R_n(x)\sin bx$.

etc... Problem with method: $f(x)$ may have components belonging to nullspace ($\ker L$).

on $V = \text{span}\{e^{rx}\}$ $L: V \to V$ is one to one and onto iff r is not root of char poly.

on $V = \text{span}\{e^{ax}\cos bx, e^{ax}\sin bx\}$ $L: V \to V$ is 1-to-1 & onto iff $a+ib$ is not root of char poly

$V = \mathbb{P}_n = \text{span}\{1, x, x^2, \dots, x^n\}$ $L: V \to V$ is 1-to-1 and onto iff. <u>ask</u> 0 is not a root. (const functions)

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<u>How to fix it?</u> trick: multiply entire ~~solution~~ guess by x^s
where s = smallest nonnegative integer so that no term in the guess belongs to ker L.

<u>Example:</u>

$$L(y) = y'' + 2y' + y$$

Solve $L(y) = e^{-x}$

$$p(r) = (r+1)^2. \qquad y_H = Ae^{-x} + Bxe^{-x}.$$

Look for $y_p = Cx^2 e^{-x}$

2 $\qquad y_p' = C(2x - x^2)e^{-x}$

$$y_p'' = C(2 - 2x - 2x + x^2)e^{-x}$$

$$L(y_p) = C(\cancel{x^2} + \cancel{4x} - \cancel{2x^2} + 2 - \cancel{4x} + \cancel{x^2})e^{-x}$$

$$= 2Ce^{-x} = e^{-x}$$

$$\Rightarrow C = \frac{1}{2}$$

$$y_p(x) = \frac{1}{2}x^2 e^{-x}.$$