

HW 9 Solutions

①

7.1.15) $f(t) = 1 + \cosh 5t \Rightarrow \boxed{F(s) = \mathcal{L}\{1\}(s) + \mathcal{L}\{\cosh 5t\}(s)}$
 $= \frac{1}{s} + \frac{1}{s^2 - 25}$

7.1.16) $f(t) = \sin 2t + \cos 2t \Rightarrow \boxed{F(s) = \mathcal{L}\{\sin 2t\}(s) + \mathcal{L}\{\cos 2t\}(s)}$
 $= \frac{2}{s^2 + 4} + \frac{s}{s^2 + 4}$

1.1.18 $f(t) = \sin 3t \cos 3t = g'(t)$ where $g(t) = \frac{1}{6} (\sin 3t)^2$
 $= \frac{1}{12} (1 - \cos 6t)$

$$G(s) = \frac{1}{12s} - \frac{1}{12} \frac{s}{s^2 + 36}$$

$$\boxed{F(s) = \mathcal{L}\{g'(t)\}(s) = sG(s) - g(0) = \frac{1}{12} \left[1 - \frac{s^2}{s^2 + 36} \right]}$$

 $= \frac{3}{s^2 + 36}$

1.1.26) $F(s) = \frac{1}{s+5} \Rightarrow \boxed{f(t) = e^{-5t}}$

7.1.28 $F(s) = \frac{3s+1}{s^2+4} = 3 \frac{s}{s^2+4} + \frac{1}{2} \frac{2}{s^2+4} = 3\mathcal{L}\{\cos 2t\}(s) + \frac{1}{2}\mathcal{L}\{\sin 2t\}(s)$

$\Rightarrow \boxed{f(t) = 3\cos 2t + \frac{1}{2}\sin 2t}$

1.1.31) $F(s) = \frac{10s-3}{25-s^2} = -\frac{10}{5} \frac{s}{s^2-25} + \frac{3}{5} \frac{5}{s^2-25} = -10\mathcal{L}\{\cosh 5t\}(s) + \frac{3}{5}\mathcal{L}\{\sinh 5t\}(s)$

$\Rightarrow \boxed{f(t) = -10 \cosh 5t + \frac{3}{5} \sinh 5t}$

7.2.2.

$$\begin{cases} x'' + 9x = 0 \\ x(0) = 3 \\ x'(0) = 4 \end{cases}$$

$$\xrightarrow{\mathcal{L}} \lambda^2 X(\lambda) - \lambda x(0) - x'(0) + 9X(\lambda) = 0$$

$$\Rightarrow (\lambda^2 + 9)X(\lambda) = 3\lambda + 4$$

$$X(\lambda) = \frac{3\lambda}{\lambda^2 + 9} + \frac{4}{\lambda^2 + 9}$$

$$\Rightarrow x(t) = 3 \cos 3t + \frac{4}{3} \sin 3t$$

7.2.4

$$\begin{cases} x'' + 8x' + 15x = 0 \\ x(0) = 2 \\ x'(0) = -3 \end{cases}$$

$$\xrightarrow{\mathcal{L}} \lambda^2 X(\lambda) + 8\lambda X(\lambda) + 15X(\lambda) =$$

$$\lambda x(0) + x'(0) + 8x(0)$$

$$\Rightarrow X(\lambda) = \frac{2\lambda + 13}{\lambda^2 + 8\lambda + 15} = \frac{2\lambda + 13}{(\lambda + 3)(\lambda + 5)}$$

$$= \frac{2(\lambda + 5)}{(\lambda + 3)(\lambda + 5)} + \frac{3}{(\lambda + 3)(\lambda + 5)}$$

$$= \frac{2}{\lambda + 3} + \frac{3}{2} \left(\frac{1}{\lambda + 3} - \frac{1}{\lambda + 5} \right)$$

$$= \frac{7}{2} \frac{1}{\lambda + 3} - \frac{3}{2} \frac{1}{\lambda + 5}$$

$$\Rightarrow x(t) = \frac{7}{2} e^{-3t} - \frac{3}{2} e^{-5t}$$

7.2.6

$$\begin{cases} x'' + 4x = \cos t \\ x(0) = x'(0) = 0 \end{cases}$$

$$\xrightarrow{\mathcal{L}} \lambda^2 X(\lambda) + 4X(\lambda) = \frac{1}{\lambda^2 + 1}$$

$$\Rightarrow X(\lambda) = \frac{1}{(\lambda^2 + 1)(\lambda^2 + 4)} = \frac{A\lambda + B}{\lambda^2 + 1} + \frac{C\lambda + D}{\lambda^2 + 4}$$

$$\Rightarrow 1 = (A\lambda + B)(\lambda^2 + 4) + (C\lambda + D)(\lambda^2 + 1)$$

$$= (A+C)\lambda^3 + (B+D)\lambda^2 + (4A+C)\lambda + 4B+D$$

$$\Rightarrow \begin{cases} A+C=0 \\ B+D=0 \\ 4A+C=1 \\ 4B+D=0 \end{cases} \Rightarrow \begin{cases} A = +\frac{1}{3} \\ B = 0 \\ C = -\frac{1}{3} \\ D = 0 \end{cases}$$

$$\Rightarrow X(\lambda) = \frac{1}{3} \frac{1}{\lambda^2 + 1} - \frac{1}{3} \frac{1}{\lambda^2 + 4}$$

$$x(t) = \frac{1}{3} [\cos t - \cos 2t]$$

7.2.13

$$\begin{cases} x' + 2y' + x = 0 \\ x' - y' + y = 0 \\ x(0) = 0; y(0) = 1 \end{cases}$$

$$\xrightarrow{\alpha} \begin{cases} \lambda X(\lambda) + 2\lambda Y(\lambda) + X(\lambda) = 2 \\ \lambda X(\lambda) - \lambda Y(\lambda) + Y(\lambda) = -1 \end{cases} \quad (3)$$

$$\begin{cases} x(t) = -\frac{2}{\sqrt{3}} \sin\left(\frac{t}{\sqrt{3}}\right) \\ y(t) = \cosh\left(\frac{t}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}} \sinh\left(\frac{t}{\sqrt{3}}\right) \end{cases}$$

$$\begin{aligned} \begin{bmatrix} \lambda+1 & 2\lambda \\ \lambda & \lambda-1 \end{bmatrix} \begin{bmatrix} X(\lambda) \\ Y(\lambda) \end{bmatrix} &= \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ \xrightarrow{\alpha^{-1}} \begin{bmatrix} X(\lambda) \\ Y(\lambda) \end{bmatrix} &= \frac{1}{1-3\lambda^2} \begin{bmatrix} \lambda-1 & -2\lambda \\ -\lambda & \lambda+1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{1-3\lambda^2} \\ \frac{-3\lambda-1}{1-3\lambda^2} \end{bmatrix} \end{aligned}$$

1.2.17

$$F(\lambda) = \frac{1}{\lambda(\lambda-3)} = \frac{1}{3} \left(\frac{1}{\lambda-3} - \frac{1}{\lambda} \right) \Rightarrow f(t) = \frac{1}{3} [e^{3t} - 1]$$

7.2.20)

$$F(\lambda) = \frac{2\lambda+1}{\lambda(\lambda^2+9)} = \frac{2}{\lambda^2+9} + \frac{1}{\lambda(\lambda^2+9)}$$

$$\begin{aligned} \frac{1}{\lambda(\lambda^2+9)} &= \frac{A}{\lambda} + \frac{C\lambda+D}{\lambda^2+9} \Rightarrow 1 = A(\lambda^2+9) + \lambda(C\lambda+D) \\ &= (A+C)\lambda^2 + D\lambda + 9A \\ \Rightarrow A &= \frac{1}{9}; C = -\frac{1}{9}; D = 0. \end{aligned}$$

$$\Rightarrow F(\lambda) = \frac{1}{9\lambda} + \frac{2 - \lambda/9}{\lambda^2+9}$$

$$\xrightarrow{\alpha^{-1}} f(t) = \frac{1}{9} + \frac{2}{3} \sin 3t - \frac{1}{9} \cos 3t$$

7.3.7

$$F(\lambda) = \frac{1}{\lambda^2+4\lambda+4} = \frac{1}{(\lambda+2)^2} \Rightarrow f(t) = e^{-2t} t$$

7.3.8)

$$F(\lambda) = \frac{\lambda+2}{\lambda^2+4\lambda+5} = \frac{\lambda+2}{(\lambda+2)^2+1} \Rightarrow f(t) = e^{-2t} \cos t$$

7.3.13)

$$F(s) = \frac{5-2s}{s^2+9s+10} = \frac{5-2s}{(s+2)(s+5)} = \frac{A}{s+2} + \frac{B}{s+5}$$

$$5-2s = (s+5)A + (s+2)B \Rightarrow \begin{cases} A+B = -2 \\ 5A+2B = 5 \end{cases} \Rightarrow A=3, B=-5$$

$$\Rightarrow F(s) = \frac{3}{s+2} - \frac{5}{s+5}$$

$$= f(t) = 3e^{-2t} - 5e^{-5t}$$

7.3.14

$$F(s) = \frac{5s-4}{s^3-s^2-2s} = \frac{5s-4}{s(s-2)(s+1)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+1}$$

$$5s-4 = A(s-2)(s+1) + Bs(s+1) + Cs(s-2)$$

$$\Leftrightarrow \begin{cases} A+B+C = 0 \\ -A+B-2C = 5 \\ -2A = -4 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=1 \\ C=-3 \end{cases}$$

$$\Rightarrow F(s) = \frac{2}{s} + \frac{1}{s-2} - \frac{3}{s+1}$$

$$\Rightarrow f(t) = 2 + e^{2t} - 3e^{-t}$$

7.3.27

$$\begin{cases} x'' + 6x' + 25x = 0 \\ x(0) = 2 \\ x'(0) = 3 \end{cases} \xrightarrow{\mathcal{L}} s^2 X(s) + 6sX(s) + 25X(s) = sx(0) + x'(0) + 6x(0) = 2s + 15$$

$$\Leftrightarrow X(s) = \frac{2s+15}{s^2+6s+25} = \frac{2s+15}{(s+3)^2+16} = \frac{2(s+3)+9}{(s+3)^2+16}$$

$$x(t) = e^{-3t} \left[2\cos 4t + \frac{9}{4}\sin 4t \right] \xleftarrow{\mathcal{L}^{-1}}$$

7.4.3

$$\begin{aligned} (\sin t * \sin t)(t) &= \int_0^t \sin z \sin(t-z) dz = \frac{1}{2} \int_0^t [-\cos t + \cos(2z-t)] dz \\ &= -\frac{1}{2}t\cos t + \frac{1}{4} \sin(2z-t) \Big|_{z=0}^t \\ &= \frac{1}{2} \sin t - \frac{1}{2}t\cos t \end{aligned}$$

$$\sin a \sin b = \frac{1}{2} [\cos(a-b) - \cos(a+b)]$$

7.4.7

$$F(s) = \frac{1}{s(s-3)} \Rightarrow f(t) = 1 * e^{3t} = \int_0^t 1 e^{3(t-z)} dz = \frac{1}{3} e^{3(t-z)} \Big|_{z=0}^t = \frac{1}{3} (1 - e^{3t})$$

7.4.10 $F(s) = \frac{1}{s^2(s^2+k^2)} = \frac{1}{k} \int_0^t f(\tau) \mathcal{L}\left\{ \frac{k}{s^2+k^2} \right\}$

$\Rightarrow f(t) = \frac{1}{k} t * \sin kt = \frac{1}{k} \int_0^t \sin k(t-z) dz$
 $= \frac{z}{k^2} \cos k(t-z) \Big|_{z=0}^t - \frac{1}{k^2} \int_0^t \cos k(t-z) dz$
 $= \frac{t}{k^2} - \frac{1}{k^3} \sin k(z-t) \Big|_{z=0}^t$
 $= \frac{t}{k^2} - \frac{1}{k^3} \sin kt$

4.15

$f(t) = t \sin 3t$ $\mathcal{L}\{t \sin 3t\}(s) = - \frac{d}{ds} \mathcal{L}\{\sin 3t\}(s)$
 $= - \frac{d}{ds} \left[\frac{3}{s^2+9} \right] = \frac{6s}{(s^2+9)^2}$

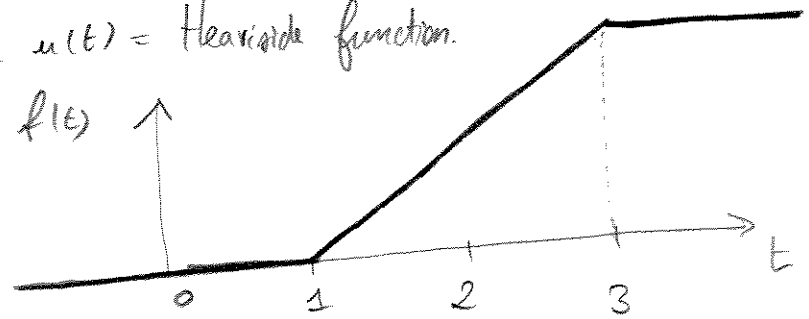
4.2)

$f(t) = \frac{e^{3t}-1}{t}$; $\mathcal{L}\{f(t)\}(s) = \int_s^\infty \mathcal{L}\{e^{3t}-1\}(\sigma) d\sigma$
 $= \int_s^\infty \left[\frac{1}{\sigma-3} - \frac{1}{\sigma} \right] d\sigma$
 $= \ln \frac{\sigma-3}{\sigma} \Big|_{\sigma=s}^\infty = 0 - \ln \frac{s-3}{s}$
 $= \ln s - \ln(s-3)$

7.5.2

$F(s) = \frac{e^{-s}-e^{-3s}}{s^2} \xrightarrow{\mathcal{L}^{-1}} f(t) = (t-1)u(t-1) - (t-3)u(t-3)$

where $u(t)$ = Heaviside function.



7.5.13

$$f(t) = \begin{cases} \sin t & \text{for } 0 \leq t \leq 2\pi \\ 0 & \text{for } t > 2\pi \end{cases}$$

sin is 2π -per.

$$f(t) = (1 - u(t - 2\pi)) \sin t = \sin t - u(t - 2\pi) \sin(t - 2\pi)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2 + 1} - \frac{e^{-2\pi s}}{s^2 + 1} = \frac{1 - e^{-2\pi s}}{s^2 + 1}$$

7.5.33

$$\begin{cases} m x'' + (r x' + k x) = f(t) \\ x(0) = x'(0) = 0 \end{cases} \quad m = 1, k = 9, c = 0$$

\mathcal{L}

$$s^2 X(s) + 9 X(s) = F(s) \stackrel{\text{from 7.5.13}}{=} \frac{1 - e^{-2\pi s}}{s^2 + 1}$$

$$\Rightarrow X(s) = \frac{1 - e^{-2\pi s}}{(s^2 + 9)(s^2 + 1)} = (1 - e^{-2\pi s}) \left(\frac{1}{s^2 + 1} - \frac{1}{s^2 + 9} \right) \frac{1}{8}$$

\mathcal{L}^{-1}

$$x(t) = \frac{1}{8} \left[\sin t - \frac{\sin 3t}{3} - u(t - 2\pi) \left(\sin(t - 2\pi) - \frac{\sin(3(t - 2\pi))}{3} \right) \right]$$

$$= \frac{1}{8} \left[\sin t - \frac{\sin 3t}{3} \right] (1 - u(t - 2\pi))$$

