HW 9 Solutions

7.1.15) $f(t) = 1 + \cosh 5t \Rightarrow \boxed{F(s) = \mathcal{L}\{1\}(s) + \mathcal{L}\{\cosh 5t\}(s) = \frac{1}{s} + \frac{s}{s^2-25}}$

7.1.16) $f(t) = \sin 2t + \cos 2t \rightarrow \boxed{F(t) = \mathcal{L}\{\sin 2t\}(s) + \mathcal{L}\{\cos 2t\}(s) = \frac{2}{s^2+4} + \frac{s}{s^2+4}}$.

7.1.18 $f(t) = \sin 3t \cos 3t = g'(t)$ where $g(t) = \frac{1}{6}(\sin 3t)^2 = \frac{1}{12}(1 - \cos 6t)$

$$G(s) = \frac{1}{12s} - \frac{1}{12}\frac{s}{(s^2+36)}$$

$$\boxed{F(s) = \mathcal{L}\{g'(t)\}(s) = s\,G(s) - \underbrace{\cancel{g(0)}}_{=0} = \frac{1}{12}\left[1 - \frac{s^2}{s^2+36}\right] = \frac{3}{s^2+36}}$$

7.1.26) $F(s) = \frac{1}{s+5} \Rightarrow \boxed{f(t) = e^{-5t}}$

7.1.28 $F(s) = \frac{3s+1}{s^2+4} = 3\frac{s}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4} = 3\mathcal{L}\{\cos 2t\}(s) + \frac{1}{2}\mathcal{L}\{\sin 2t\}(s)$

$\rightarrow \boxed{f(t) = 3\cos 2t + \frac{1}{2}\sin 2t}$

7.1.31) $F(s) = \frac{10s-3}{25-s^2} = -10\frac{s}{s^2-25} + \frac{3}{5}\frac{5}{s^2-25} = -10\mathcal{L}\{\cosh 5t\}(s) + \frac{3}{5}\mathcal{L}\{\sinh 5t\}(s)$

$\Rightarrow \boxed{f(t) = -10\cosh 5t + \frac{3}{5}\sinh 5t}$

7.2.2

$$\begin{cases} x'' + 9x = 0 \\ x(0) = 3 \\ x'(0) = 4 \end{cases} \xrightarrow{\mathcal{L}} s^2 X(s) - s\underbrace{x(0)}_{3} - \underbrace{x'(0)}_{4} + 9X(s) = 0$$

$$\Leftrightarrow (s^2+9)X(s) = 3s+4$$

$$X(s) = \frac{3s}{s^2+9} + \frac{4}{s^2+9}$$

$$\Rightarrow \boxed{x(t) = 3\cos 3t + \frac{4}{3}\sin 3t}$$

7.2.4

$$\begin{cases} x'' + 8x' + 15x = 0 \\ x(0) = 2 \\ x'(0) = -3 \end{cases} \xrightarrow{\mathcal{L}} s^2 X(s) + 8sX(s) + 15X(s) = s\underbrace{x(0)}_{2} + \underbrace{x'(0)}_{-3} + 8\underbrace{x(0)}_{2}$$

$$\Rightarrow X(s) = \frac{2s+13}{s^2+8s+15} = \frac{2s+13}{(s+3)(s+5)}$$

$$= \frac{2(s+5)}{(s+3)(s+5)} + \frac{3}{(s+3)(s+5)}$$

$$= \frac{2}{s+3} + \frac{3}{2}\left(\frac{1}{s+3} - \frac{1}{s+5}\right)$$

$$= \frac{7}{2}\frac{1}{s+3} - \frac{3}{2}\frac{1}{s+5}$$

$$\xrightarrow{\mathcal{L}^{-1}} \Rightarrow \boxed{x(t) = \frac{7}{2}e^{-3t} - \frac{3}{2}e^{-5t}}$$

7.2.6

$$\begin{cases} x'' + 4x = \cos t \\ x(0) = x'(0) = 0 \end{cases} \xrightarrow{\mathcal{L}} s^2 X(s) + 4X(s) = \frac{s}{s^2+1}$$

$$\Rightarrow X(s) = \frac{s}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$\Leftrightarrow s = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

$$= (A+C)s^3 + (B+D)s^2 + (4A+C)s + 4B+D$$

$$\Rightarrow \begin{cases} A+C = 0 \\ B+D = 0 \\ 4A+C = 1 \\ 4B+D = 0 \end{cases} \rightsquigarrow \begin{cases} A = +\frac{1}{3} \\ B = 0 \\ C = -\frac{1}{3} \\ D = 0 \end{cases}$$

$$\Rightarrow X(s) = \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4}$$

$$\xrightarrow{\mathcal{L}^{-1}} \boxed{x(t) = \frac{1}{3}\left[\cos t - \cos 2t\right]}$$

7.2.13
$$\begin{cases} x' + 2y' + x = 0 \\ x' - y' + y = 0 \\ x(0)=0;\ y(0)=1 \end{cases} \xrightarrow{\mathcal{L}} \begin{cases} sX(s) + 2sY(s) + X(s) = 2 \\ sX(s) - sY(s) + Y(s) = -1 \end{cases}$$

(3)

$$\begin{bmatrix} s+1 & 2s \\ s & 1-s \end{bmatrix}\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \frac{1}{1-3s^2}\begin{bmatrix} 1-s & -2s \\ -s & s+1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{2}{1-3s^2} \\ \frac{-3s-1}{1-3s^2} \end{bmatrix}$$

$\xleftarrow{\mathcal{L}^{-1}}$

$$\boxed{\begin{aligned} x(t) &= -\frac{2}{\sqrt{3}}\sinh(t/\sqrt{3}) \\ y(t) &= \cosh\left(\frac{t}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}}\sinh\left(\frac{t}{\sqrt{3}}\right) \end{aligned}}$$

1.2.17
$$F(s) = \frac{1}{s(s-3)} = \frac{1}{3}\left(\frac{1}{s-3} - \frac{1}{s}\right) \Rightarrow f(t) = \frac{1}{3}\left[e^{3t} - 1\right]$$

7.2.20)
$$F(s) = \frac{2s+1}{s(s^2+9)} = \frac{2}{s^2+9} + \frac{1}{s(s^2+9)}$$

$$\frac{1}{s(s^2+9)} = \frac{A}{s} + \frac{Cs+D}{s^2+9} \Rightarrow 1 = A(s^2+9) + s(Cs+D)$$
$$= (A+C)s^2 + Ds + 9A$$
$$\Rightarrow A = \frac{1}{9};\ C = -\frac{1}{9};\ D = 0$$

$$\Rightarrow F(s) = \frac{1}{9}\frac{1}{s} + \frac{2 - s/9}{s^2+9}$$

$$\xrightarrow{\mathcal{L}^{-1}} \boxed{f(t) = \frac{1}{9} + \frac{2}{3}\sin 3t - \frac{1}{9}\cos 3t}$$

7.3.7
$$F(s) = \frac{1}{s^2+4s+4} = \frac{1}{(s+2)^2} \Rightarrow \boxed{f(t) = e^{-2t}\,t}$$

7.3.8)
$$F(s) = \frac{s+2}{s^2+4s+5} = \frac{s+2}{(s+2)^2+1} \Rightarrow \boxed{f(t) = e^{-2t}\cos t}$$

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7.3.13) $F(s) = \dfrac{5-2s}{s^2+7s+10} = \dfrac{5-2s}{(s+2)(s+5)} = \dfrac{A}{s+2} + \dfrac{B}{s+5}$

$5-2s = (s+5)A + (s+2)B \Rightarrow \begin{cases} A+B = -2 \\ 5A+2B = 5 \end{cases}$

$\Rightarrow A = 3,\ B = -5$

$\Rightarrow F(s) = \dfrac{3}{s+2} - \dfrac{5}{s+5}$

$$\boxed{\Rightarrow f(t) = 3e^{-2t} - 5e^{-5t}}$$

7.3.14 $F(s) = \dfrac{5s-4}{s^3-s^2-2s} = \dfrac{5s-4}{s(s-2)(s+1)} = \dfrac{A}{s} + \dfrac{B}{s-2} + \dfrac{C}{s+1}$

$5s-4 = A(s-2)(s+1) + Bs(s+1) + Cs(s-2)$

$\Leftrightarrow \begin{cases} A+B+C = 0 \\ -A+B-2C = 5 \\ -2A = -4 \end{cases} \Rightarrow \begin{array}{l} A = 2 \\ B = 1 \\ C = -3 \end{array}$

$\Rightarrow F(s) = \dfrac{2}{s} + \dfrac{1}{s-2} - \dfrac{3}{s+1}$

$$\Rightarrow \boxed{f(t) = 2 + e^{2t} - 3e^{-t}}$$

7.3.27 $\begin{cases} x'' + 6x' + 25x = 0 \\ x(0) = 2 \\ x'(0) = 3 \end{cases} \xrightarrow{\mathcal{L}} s^2X(s) + 6sX(s) + 25X(s) = sx(0) + x'(0) + 6x(0) = 2s + 15$

$\Rightarrow X(s) = \dfrac{2s+15}{s^2+6s+25} = \dfrac{2s+15}{(s+3)^2+16} = \dfrac{2(s+3)+9}{(s+3)^2+16}$

$$\boxed{x(t) = e^{-3t}\left[2\cos 4t + \frac{9}{4}\sin 4t\right]} \xleftarrow{\mathcal{L}^{-1}}$$

7.4.3 $\boxed{(\sin t * \sin t)(t)} = \displaystyle\int_0^t \sin \tau \sin(t-\tau)\,d\tau = \frac{1}{2}\int_0^t [-\cos t + \cos(2\tau - t)]\,d\tau$

$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$

$= -\frac{1}{2}t\cos t + \frac{1}{4}\sin(2\tau - t)\Big|_{\tau=0}^{t}$

$$= \boxed{\frac{1}{2}\sin t - \frac{1}{2}t\cos t}$$

7.4.7 $F(s) = \dfrac{1}{s(s-3)} \Rightarrow f(t) = 1 * e^{3t} = \displaystyle\int_0^t 1\, e^{3(t-\tau)}\,d\tau$

$= \frac{1}{3}e^{3(t-\tau)}\Big|_{\tau=0}^{t} = \boxed{\frac{1}{3}(1 - e^{3t})}$

7.4.10

$$F(s) = \frac{1}{s^2(s^2+k^2)} = \frac{1}{k}\mathcal{L}\{t\}(s)\,\mathcal{L}\left\{\frac{k}{s^2+k^2}\right\}$$

$$\Rightarrow \boxed{f(t) = \frac{1}{k}\, t * \sin kt} = \frac{1}{k}\int_0^t \tau \sin k(t-\tau)\,d\tau$$

$$= \frac{\tau}{k^2}\cos k(t-\tau)\Big|_{\tau=0}^{t} - \frac{1}{k^2}\int_0^t \cos k(t-\tau)\,d\tau$$

$$= \frac{t}{k^2} - \frac{1}{k^3}\sin k(\tau - t)\Big|_{\tau=0}^{t}$$

$$\boxed{= \frac{t}{k^2} - \frac{1}{k^3}\sin kt}$$

7.4.15

$$f(t) = t\sin 3t \qquad \mathcal{L}\{t\sin 3t\}(s) = -\frac{d}{ds}\mathcal{L}\{\sin 3t\}(s)$$

$$\boxed{= -\frac{d}{ds}\left[\frac{3}{s^2+9}\right] = \frac{6s}{(s^2+9)^2}}$$

7.4.21

$$f(t) = \frac{e^{3t}-1}{t}; \quad \mathcal{L}\{f(t)\}(s) = \int_s^\infty \mathcal{L}\{e^{3t}-1\}(\sigma)\,d\sigma$$

$$= \int_s^\infty \left[\frac{1}{\sigma-3} - \frac{1}{\sigma}\right]d\sigma$$

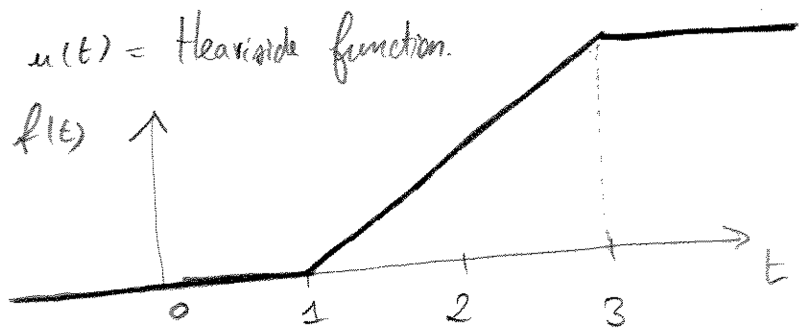
$$= \ln\frac{\sigma-3}{\sigma}\Big|_{\sigma=s}^{\infty} = 0 - \ln\frac{s-3}{s}$$

$$\boxed{= \ln s - \ln(s-3).}$$

7.5.2

$$F(s) = \frac{e^{-s} - e^{-3s}}{s^2} \xrightarrow{\mathcal{L}^{-1}} f(t) = (t-1)u(t-1) - (t-3)u(t-3)$$

where $u(t)$ = Heaviside function.



7.5.13

$$f(t) = \begin{cases} \sin t & \text{for } 0 \le t \le 2\pi \\ 0 & \text{for } t > 2\pi \end{cases}$$

sin is 2π-per.

$$f(t) = (1 - u(t-2\pi))\sin t = \sin t - u(t-2\pi)\sin(t-2\pi)$$

$$\xrightarrow{\mathcal{L}} \boxed{F(s) = \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1} = \frac{1-e^{-2\pi s}}{s^2+1}}$$

7.5.33 $\begin{cases} m x'' + c x' + k x = f(t) \\ x(0) = x'(0) = 0 \end{cases}$ $m = 1, k = 9, c = 0$

$\downarrow \mathcal{L}$ from 7.5.13

$$s^2 X(s) + 9 X(s) = F(s) = \frac{1-e^{-2\pi s}}{s^2+1}$$

$$\Rightarrow X(s) = \frac{1-e^{-2\pi s}}{(s^2+9)(s^2+1)} = (1-e^{-2\pi s})\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)\frac{1}{8}$$

$$\xrightarrow{\mathcal{L}^{-1}} x(t) = \frac{1}{8}\left[\sin t - \frac{\sin 3t}{3} - u(t-2\pi)\left(\sin(t-\cancel{2\pi}) - \frac{\sin(3(t-\cancel{2\pi}))}{3}\right)\right]$$

$$= \frac{1}{8}\left[\sin t - \frac{\sin 3t}{3}\right](1 - u(t-2\pi))$$