Homework 8 Solutions

6.1.6
$$\begin{cases} \dfrac{dx}{dt} = 2 - 4x - 15y \\ \dfrac{dy}{dt} = 4 - x^2 \end{cases}$$

critical points:
$$\begin{cases} 2 - 4x - 15y = 0 \\ \text{and } 4 - x^2 = 0 \end{cases} \Rightarrow \begin{array}{l} x = 2 \text{ and } y = -2/5 \\ \text{or} \\ x = -2 \text{ and } y = 2/3 \end{array}$$

$\to$ $\boxed{\text{fig } 6.1.18}$ since $(2, -2/5)$ and $(-2, 2/3)$ are crit pts.

6.1.7
$$\begin{cases} \dfrac{dx}{dt} = x - 2y \\ \dfrac{dy}{dt} = 4x - x^3 \end{cases}$$

critical pts:
$$\begin{cases} x - 2y = 0 \\ \text{and } 4x - x^3 = 0 \end{cases} \Rightarrow \begin{cases} x = 2y \\ \text{and } x(4 - x^2) = 0 \end{cases}$$

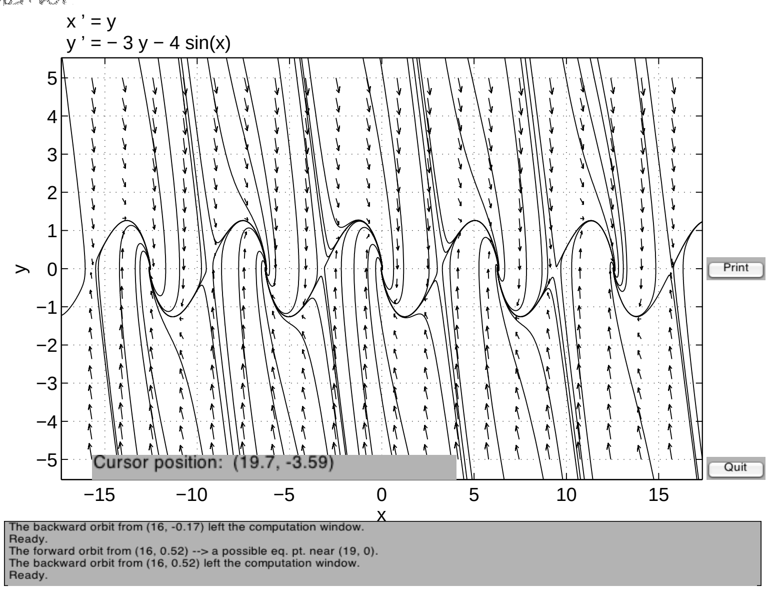
$\Rightarrow$ $x = 0,\ y = 0$ or $x = 2,\ y = 1$; $x = -2,\ y = -1$ $\Rightarrow$ $(0,0)$, $(2,1)$, $(-2,-1)$ are critical points.

corresponds to fig. 6.1.12.

6.1.11
$$x'' + 3x' + 4\sin x = 0 \iff \begin{cases} x' = y \\ y' = -3y - 4\sin x \end{cases}$$

critical pts are: $\begin{cases} y = 0 \\ \sin x = 0 \end{cases} \iff (k\pi, 0),\ k \in \mathbb{Z}.$

k even spiral
k odd saddle point

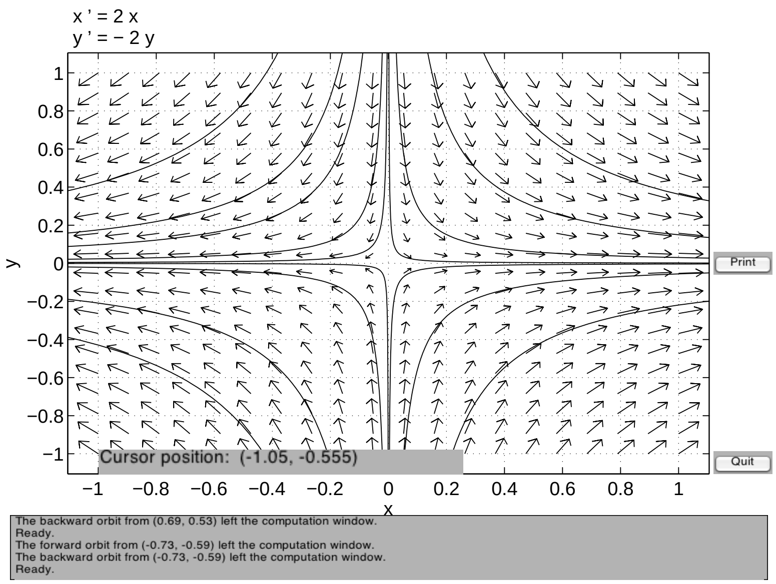


6.1.14

$$\begin{cases} \dfrac{dx}{dt} = 2x \\ \dfrac{dy}{dt} = -2y \end{cases} \Rightarrow \text{critical pt: } \begin{cases} 2x = 0 \\ -2y = 0 \end{cases} \Rightarrow (0,0) \text{ is the only critical pt.}$$

$$\Leftrightarrow \quad \underline{x}' = A\underline{x} \quad \text{where } A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

eigenvalues have opposite signs
$\Rightarrow$ <u>saddle pt.</u> (unstable)

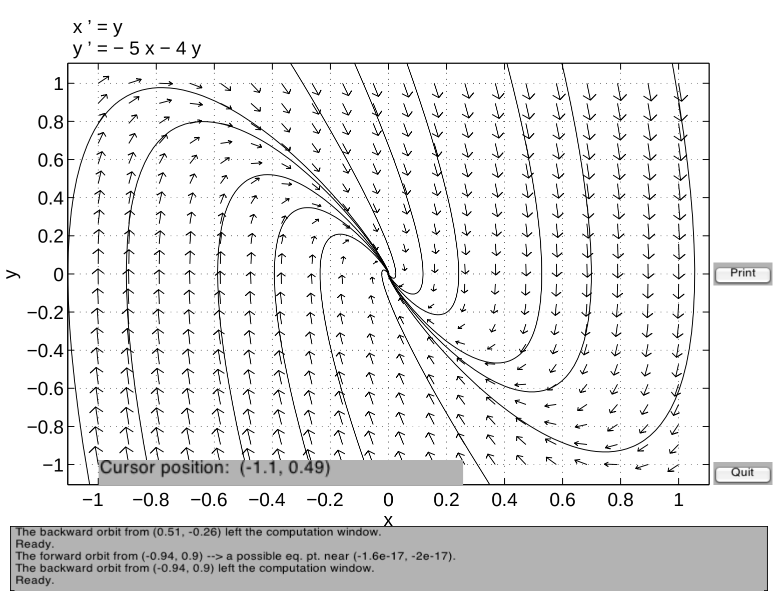


6.1.20

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -5x - 4y \end{cases} \qquad \text{critical pts: } \begin{cases} y = 0 \\ -5x - 4y = 0 \end{cases} \Rightarrow (0,0) \text{ is the only crit. pt.}$$

$$\Leftrightarrow \quad \underline{x}' = A\underline{x} \quad \text{where } A = \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix} \quad \text{eigenvalues are: } \lambda = -2 \pm i$$

$\Rightarrow$ spiral, <u>stable</u> since $\mathrm{Re}(\lambda) < 0$.



6.1.24

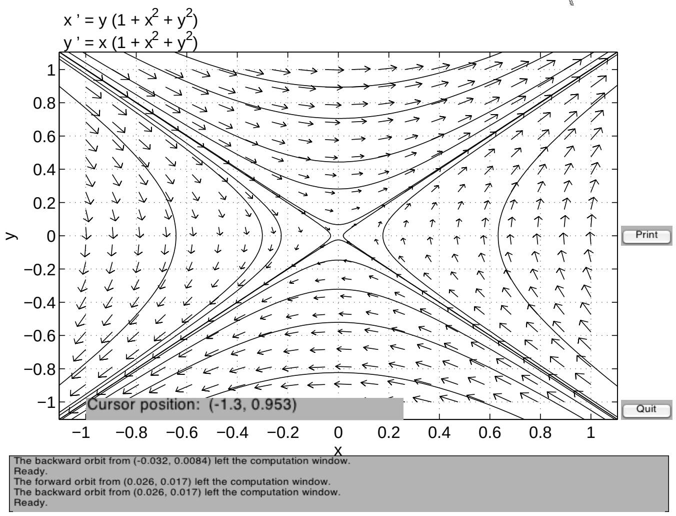
$$\begin{cases} \dfrac{dx}{dt} = y\,(1+x^2+y^2) \\[2mm] \dfrac{dy}{dt} = x\,(1+x^2+y^2) \end{cases}$$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{x}{y}$.

$\Rightarrow \int y\,dy = \int x\,dx + c$

$\Rightarrow y^2 = x^2 + c$

trajectories are hyperbolas, so we must have a saddle point (unstable).



(6.2.5) $\underline{x}' = A\underline{x}$ where $A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$

$p(\lambda) = (1-\lambda)(-3-\lambda) + 4$
$= \lambda^2 + 2\lambda + 1 = (\lambda+1)^2$

$\lambda = -1$ is an eigenvalue of algebraic multiplicity 2.
Since $\lambda < 0$, $(0,0)$ is a stable crit pt. It's either a star or an improper node depending on the geometric multiplicity (dim ker $A + I$):

$(A+I)\underline{v} = \underline{0} \Leftrightarrow \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and thus: dim ker $A+I = 1$.

$\Rightarrow$ improper node (see phase portrait in book).

6.2.6 $\underline{x}' = A\underline{x}$ with $A = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}$

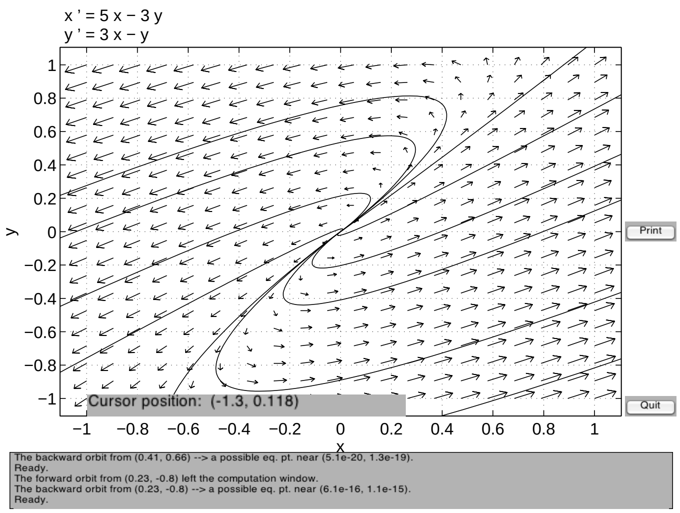
$$p(\lambda) = (5-\lambda)(-1-\lambda)+9 = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$$

$\Rightarrow \lambda = 2$ is an eigenvalue with algebraic multiplicity 2.

eigenvectors: $(A - 2I)\underline{v} = \underline{0} \Leftrightarrow \begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} c \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow \dim \ker A - 2I = 1$

defective eigenvalue $\lambda = 2 \Rightarrow$ improper unstable mode.
since $\operatorname{Re}\lambda > 0$.



(6.2.7) $\underline{x}' = A\underline{x}$ with $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$

$$p(\lambda) = (3-\lambda)(-1-\lambda)+8 = \lambda^2 - 2\lambda + 5 = (\lambda-(1+2i))(\lambda-(1-2i))$$

$\lambda = 1 \pm 2i$ are eigenvalues. We have $\operatorname{Re}\lambda > 0 \Rightarrow$ unstable spiral.

see book for phase portrait

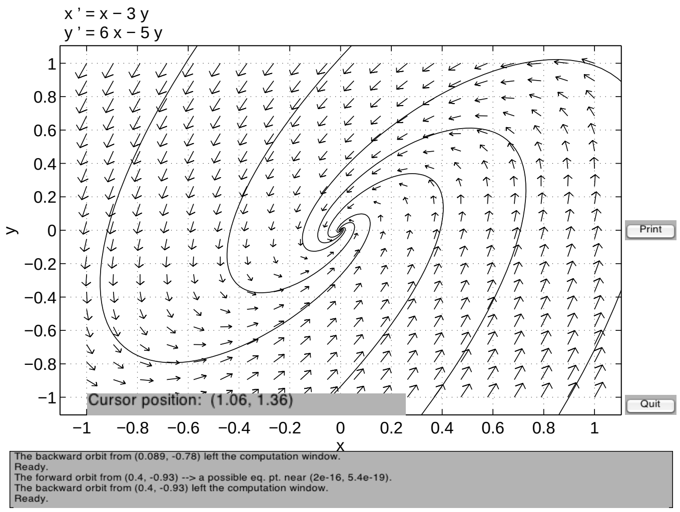
6.2.8 $\underline{x}' = A\underline{x}$ with $A = \begin{bmatrix} 1 & -3 \\ 6 & -5 \end{bmatrix}$

$$P(\lambda) = (1-\lambda)(-5-\lambda)+18 = \lambda^2 + 4\lambda + 13 = (\lambda-(-2+3i))(\lambda-(-2-3i))$$

$\lambda = -2 \pm 3i$ are the eigenvalues of A.

We have $\operatorname{Re}\lambda < 0 \Rightarrow$ stable spiral

6.2.8 (cont'd)



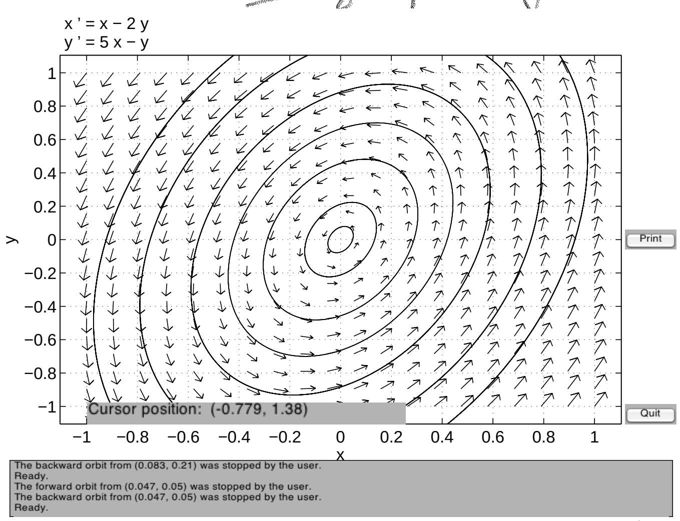
6.2.10

$\underline{x}' = A\underline{x}$ with $A = \begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix}$; $p(\lambda) = (1-\lambda)(-1-\lambda) + 10$

$= \lambda^2 + 9$

$= (\lambda - 3i)(\lambda + 3i)$

Eigenvalues of A are purely imaginary: $\lambda = \pm 3i$.
Thus we have a stable center (not asymptotically stable).



(6.2.12)

The system is of the form $\underline{x}' = A\underline{x} + \underline{b}$

with $A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$ and $\underline{b} = \begin{bmatrix} -8 \\ 10 \end{bmatrix}$

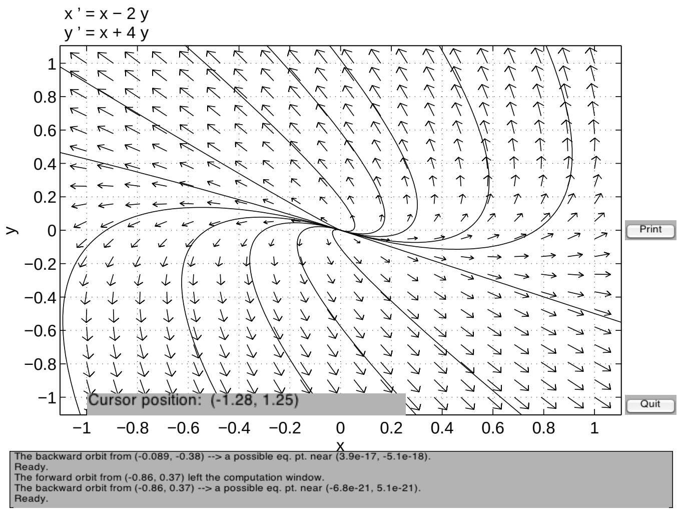
$\Rightarrow$ critical pt $\underline{x}^*$ satisfies: $A\underline{x}^* = -\underline{b} \Rightarrow \underline{x}^* = -A^{-1}\underline{b} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

Obviously Jacobian of system is A, thus we need to study its eigenvalues/eigenvectors:

$p(\lambda) = (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$.

$\Rightarrow$ Both eigenvalues have positive real part $\Rightarrow$ unstable node.

(6.2.12) Cont'd



(6.2.16) The system is of the form: $\underline{x}' = A\underline{x} + \underline{b}$

where $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$ and $\underline{b} = \begin{bmatrix} 1 \\ -9 \end{bmatrix}$ $\Rightarrow$ critical point $\underline{x}^*$ is:

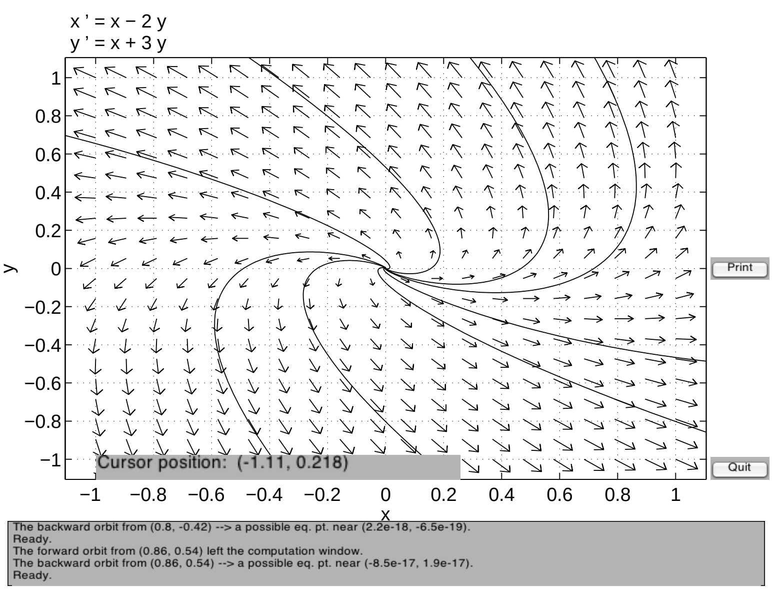
$$\underline{x}^* = -A^{-1}\underline{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

the Jacobian of the system is A.

$p(\lambda) = \det(A - \lambda I) = (1-\lambda)(3-\lambda) + 2$

$= \lambda^2 - 4\lambda + 5 = (\lambda - (2+i))(\lambda - (2-i)) \Rightarrow \lambda = 2 \pm i$

$\Rightarrow$ Since $\mathrm{Re}(\lambda) > 0$, this is an unstable spiral.



6.2.25

$$\begin{cases} \dfrac{dx}{dt} = x - 2y + 3xy \\ \dfrac{dy}{dt} = 2x - 3y - x^2 - y^2 \end{cases}$$

It is easy to verify that $(0,0)$ is a critical point.

$$J(x,y) = \begin{bmatrix} 1+3y & -2+3x \\ 2-2x & -3-2y \end{bmatrix}$$

$$\Rightarrow \quad J(0,0) = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}. \quad p(\lambda) = (1-\lambda)(-3-\lambda) + 4 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2.$$

$\lambda = 1$ is an eigenvalue of algebraic multiplicity 2.
Since $\lambda > 0$ this an unstable point. Since the eigenvalue is repeated this can either be an unstable node or unstable spiral.
(the linearization does not suffice to distinguish between these two when the eigenvalue is repeated).
In the phase portrait (see book) the origin $(0,0)$ appears to be an unstable node

6.2.28

$$\begin{cases} \dfrac{dx}{dt} = 3x - y + x^3 + y^3 \\ \dfrac{dy}{dt} = 13x - 3y + 3xy \end{cases}$$

It is easy to verify that $(0,0)$ is a critical point.

$$J(x,y) = \begin{bmatrix} 3+3x^2 & -1+3y^2 \\ 13+3y & -3+3x \end{bmatrix}$$

$$\Rightarrow \quad J(0,0) = \begin{bmatrix} 3 & -1 \\ 13 & -3 \end{bmatrix}. \quad p(\lambda) = (3-\lambda)(-3-\lambda) + 13 = \lambda^2 + 4 = (\lambda - 2i)(\lambda + 2i)$$

Since the eigenvalues $\lambda = \pm 2i$ are purely imaginary, the equilibrium of the linearized problem is a stable center.
However we cannot conclude from the linearization (study of Jacobian) on the stability of the critical point $(0,0)$.
The phase portrait does show $(0,0)$ is a stable center.

6.2.31

$$\begin{cases} \dfrac{dx}{dt} = y^2 - 1 \\ \dfrac{dy}{dt} = x^3 - y \end{cases}$$

critical pts satisfy:

$$\begin{cases} y^2 - 1 = 0 \\ x^3 - y = 0 \end{cases} \Rightarrow$$

$(1, 1)$ and $(-1, -1)$ are critical pts

6.2.31 (cont'd) stability study: We start by computing the Jacobian:

$$J(x,y) = \begin{bmatrix} 0 & 2y \\ 3x^2 & -1 \end{bmatrix}$$

For (1,1):

$$J(1,1) = \begin{bmatrix} 0 & 2 \\ 3 & -1 \end{bmatrix} \qquad p(\lambda) = (-\lambda)(-1-\lambda) - 6$$
$$= \lambda^2 + \lambda - 6 = (\lambda-2)(\lambda+3)$$

$\lambda = 2$ and $\lambda = -3$ are the eigenvalues. They are of different sign $\Rightarrow$ crit pt is an (unstable) saddle point. (here the critical pt remains of the same nature as that of the linearized problem)

For (-1,-1):

$$J(-1,-1) = \begin{bmatrix} 0 & -2 \\ 3 & -1 \end{bmatrix} \qquad p(\lambda) = (-\lambda)(-1-\lambda) + 6 = \lambda^2 + \lambda + 6$$

$$\lambda = \frac{-1 \pm \sqrt{1-24}}{2} = -\frac{1 \pm i\sqrt{23}}{2}$$

On this critical pt the Jacobian has complex conjugate eigenvalues with negative real part. Thus the critical point is a stable spiral.

(again here the nature of the critical pt is revealed by the linearization)

See phase portrait in book.

6.3.8 In problems 6.3.8 – 6.3.10 we consider the non linear system of DEs modeling competition between two species.

$$\begin{cases} \dfrac{dx}{dt} = 60x - 3x^2 - 4xy \\[2ex] \dfrac{dy}{dt} = 42y - 3y^2 - 2xy \end{cases}$$

It is useful to compute the Jacobian of this system:

$$J(x,y) = \begin{bmatrix} 60 - 6x - 4y & -4x \\ -2y & 42 - 6y - 2x \end{bmatrix}$$

6.3.8 (cont'd) at (0, 14) the Jacobian is:

$$J(0,14) = \begin{bmatrix} 4 & 0 \\ -28 & -42 \end{bmatrix}$$

(consistent with formulation in book)

$p(\lambda) = (4-\lambda)(-42-\lambda)$

$\to$ 2 eigenvalues (real) $\lambda = 4$ and $\lambda = -42$

of different signs $\Rightarrow$ saddle point.

(6.3.9) We only need to evaluate Jacobian at (20, 0) and study its eigenvalues.

$$J(20,0) = \begin{bmatrix} -60 & -80 \\ 0 & 2 \end{bmatrix}$$

$p(\lambda) = (-60-\lambda)(2-\lambda)$

$\to$ 2 eigenvalues $\lambda = -60$ and $\lambda = 2$

of different signs $\Rightarrow$ saddle point

6.3.10 At the critical point (12, 6) we have:

$$J(12,6) = \begin{bmatrix} -36 & -48 \\ -12 & -18 \end{bmatrix}$$

$$p(\lambda) = (-36-\lambda)(-18-\lambda) - 576$$
$$= \lambda^2 + 54\lambda + 72$$

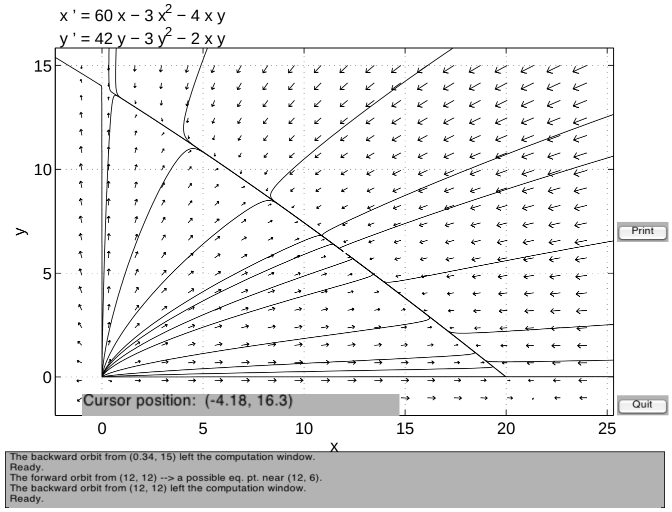
roots are $\lambda = \dfrac{-54 \pm \sqrt{2916 - 288}}{2}$

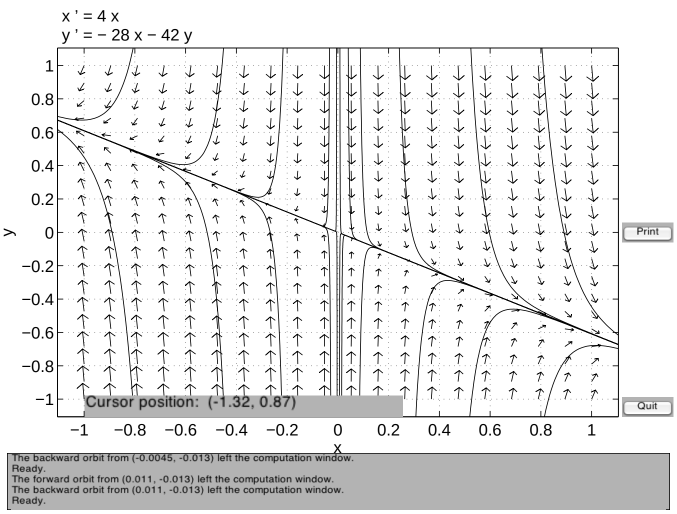
$$= -27 \pm 3\sqrt{73}$$

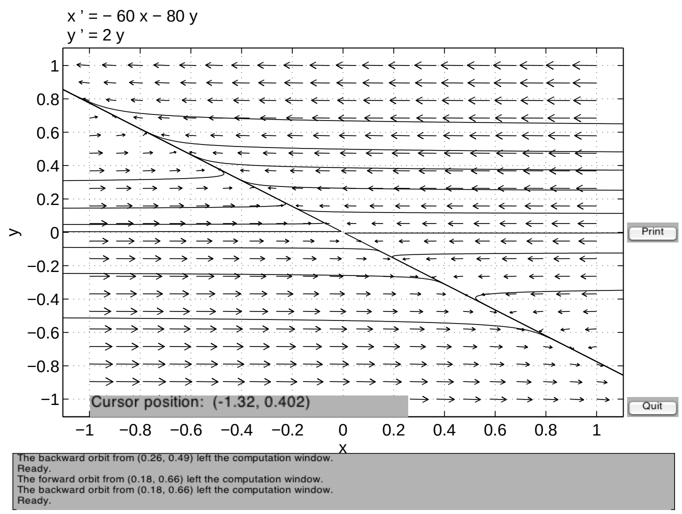
Both roots are negative $\Rightarrow$ stable node. for the NL system

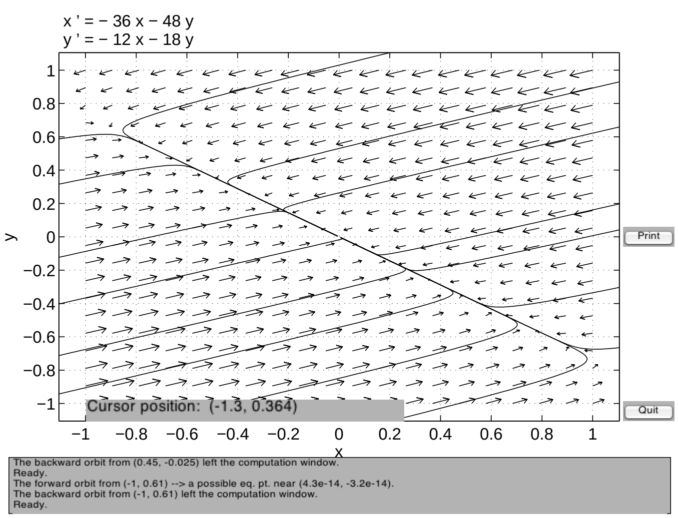
See plots on next page.

plots for 6.3.8 - 6.3.10









6.3.14 Problem 6.3.14 - 6.3.17 deal with the non-linear system

$$\begin{cases} \dfrac{dx}{dt} = x^2 - 2x - xy \\ \dfrac{dy}{dt} = y^2 - 4y + xy \end{cases}$$

The Jacobian of this system is $J(x,y) = \begin{bmatrix} 2x-2-y & -x \\ y & 2y-4+x \end{bmatrix}$

At $(0,0)$: $J(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$ $\rightsquigarrow$ eigenvalues are $\lambda = -2$, $\lambda = -4$.
$\rightsquigarrow$ stable node for non linear problem

(6.3.15) At $(0,4)$:

$J(0,4) = \begin{bmatrix} -6 & 0 \\ 4 & 4 \end{bmatrix}$; $p(\lambda) = (-6-\lambda)(4-\lambda)$
$\Rightarrow \lambda = -6, \lambda = 4$ are eigenvalues
They are real w/ opposite signs
$\Rightarrow$ $(0,4)$ is an (unstable) saddle pt for N.L. system

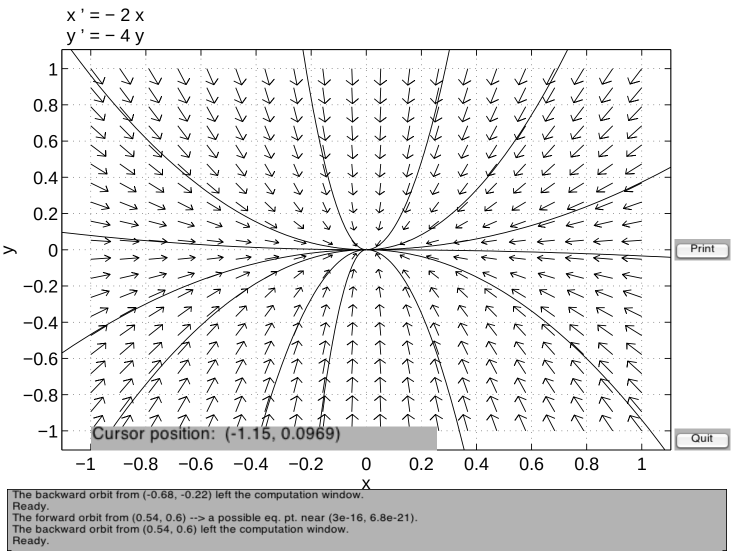
6.3.16 At $(2,0)$:

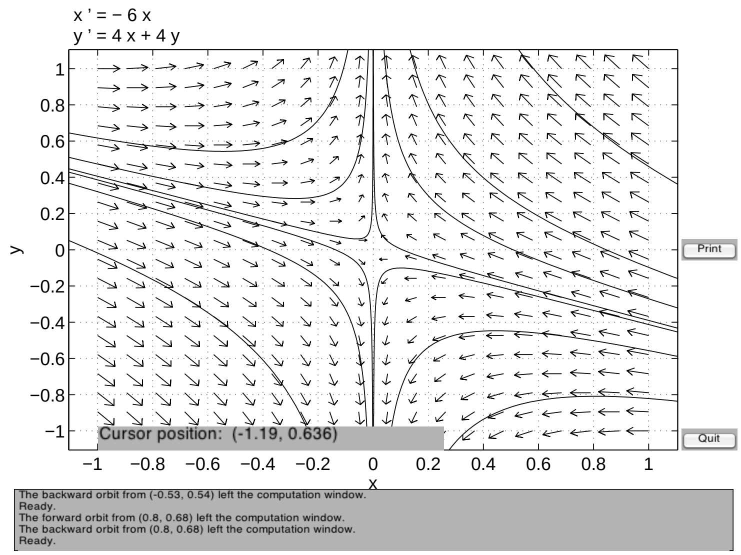
$J(2,0) = \begin{bmatrix} 2 & -2 \\ 0 & -2 \end{bmatrix}$; $p(\lambda) = (2-\lambda)(-2-\lambda)$
$\Rightarrow \lambda = 2$, $\lambda = -2$ are eigenvalues.
They have opposite signs
$\Rightarrow (2,0)$ is an (unstable) saddle pt

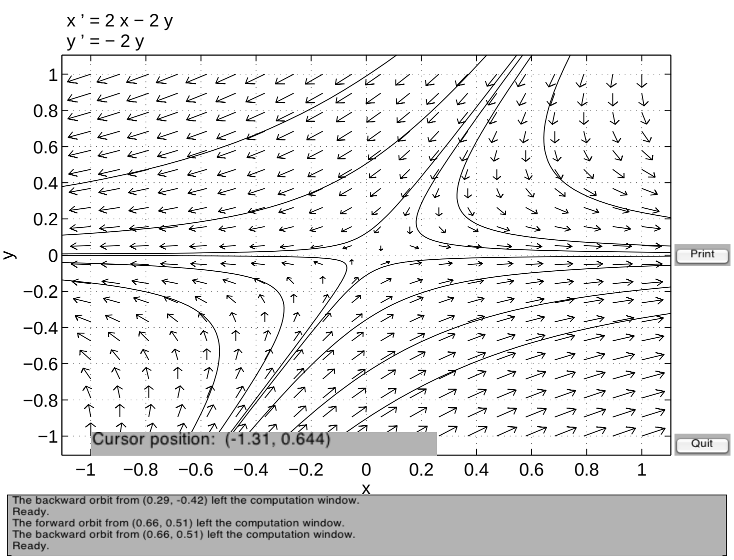
6.3.17 At $(3,1)$: $J(3,1) = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$; $p(\lambda) = (3-\lambda)(1-\lambda) + 3$
$= \lambda^2 - 4\lambda + 6$
$\Rightarrow \lambda = \dfrac{4 \pm \sqrt{16-24}}{2}$
$= 2 \pm i\sqrt{2}$

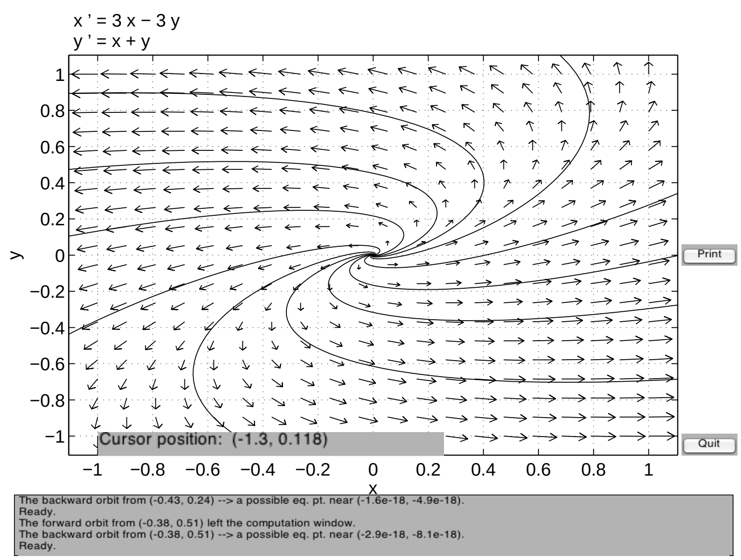
eigenvalues are complex conjugate with positive real part $\Rightarrow$ $(3,1)$ is an unstable spiral.

plots for 6.3.14 – 6.3.17









(6.4.12) $x'' + 20x - 5x^3 = 0$

$$\Leftrightarrow \begin{cases} x' = y \\ y' = -20x + 5x^3 \end{cases}$$

critical points : $\cdot\ y = 0$

$$-20x + 5x^3 = 0$$
$$= 5x(-4 + x^2)$$
$$= 5x(x-2)(x+2)$$

$\Rightarrow$ critical points are $(0,0)$, $(2,0)$ and $(-2,0)$.

Linearization study : $J(x,y) = \begin{bmatrix} 0 & 1 \\ -20+15x^2 & 0 \end{bmatrix}$

At $(0,0)$:

$J(0,0) = \begin{bmatrix} 0 & 1 \\ -20 & 0 \end{bmatrix}$; $p(\lambda) = \lambda^2 + 20 \Rightarrow \lambda = \pm i\sqrt{20}$

$\leadsto$ stable center. However linearization in this case does not guarantee stability nor instability of the critical point. This system is one of the cases for which the NL. problem has a stable center as well, according to Fig 6.4.4.

At $(2,0)$ and $(-2,0)$: $J(2,0) = J(-2,0) = \begin{bmatrix} 0 & 1 \\ 40 & 0 \end{bmatrix}$; $p(\lambda) = \lambda^2 - 40$

$\Rightarrow \lambda = \pm 2\sqrt{10}$

$\leadsto$ two real eigenvalues with different signs

$\leadsto$ $(2,0)$ and $(-2,0)$ are unstable saddle pts.

(consistent with Fig 6.4.4).

6.4.13 $x'' + 2x' + 20x - 5x^3 = 0$

$$\Leftrightarrow \begin{cases} x' = y \\ y' = -20x - 2y + 5x^3 \end{cases}$$

crit pts: $y = 0$ and $-20x + 5x^3 = 0$

$\Rightarrow$ crit. pts are: $(0,0)$, $(2,0)$ and $(-2,0)$.

Linearization study: $J(x,y) = \begin{bmatrix} 0 & 1 \\ -20+15x^2 & -2 \end{bmatrix}$

6.4.13 (cont'd)

At (0,0): $J(0,0) = \begin{bmatrix} 0 & 1 \\ -20 & -2 \end{bmatrix}$; $p(\lambda) = (-\lambda)(-2-\lambda) + 20$

$$= \lambda^2 + 2\lambda + 20$$

$$\Rightarrow \lambda = \frac{-2 \pm \sqrt{4-80}}{2}$$

$$= -1 \pm i\sqrt{19}$$

$\leadsto$ stable spiral (cplx conj. pair of eigenvals with negative real part)

(see fig 6.4.6).

At (2,0) and (-2,0):

$J(2,0) = J(-2,0) = \begin{bmatrix} 0 & 1 \\ 40 & -2 \end{bmatrix}$; $p(\lambda) = (-\lambda)(-2-\lambda) - 40$

$$= \lambda^2 + 2\lambda - 40$$

$$\Rightarrow \lambda = \frac{-2 \pm \sqrt{4+160}}{2}$$

$$= -1 \pm \sqrt{41}$$

(opposite sign real eigenvalues)

$\leadsto$ (2,0) and (-2,0) are unstable saddle points.

(consistent with fig 6.4.6)

(6.4.14) $x'' - 8x + 2x^3 = 0 \iff \begin{cases} x' = y \\ y' = 8x - 2x^3 \end{cases}$

crit pts: $\begin{cases} y = 0 \\ \text{and} \\ 2x(4-x^2) = 0 \end{cases}$

$\Rightarrow$ crit pts are

$\boxed{(0,0),\ (2,0),\ (-2,0)}$

Linearization study:

$J(x,y) = \begin{bmatrix} 0 & 1 \\ 8-6x^2 & 0 \end{bmatrix}$

At (0,0): $J(0,0) = \begin{bmatrix} 0 & 1 \\ 8 & 0 \end{bmatrix}$; $p(\lambda) = (-\lambda)^2 - 8 \Rightarrow \lambda = \pm 2\sqrt{2}$

$\leadsto$ saddle point (unstable) see fig. 6.4.12

At (2,0) and (-2,0): $J(2,0) = J(-2,0) = \begin{bmatrix} 0 & 1 \\ -16 & 0 \end{bmatrix}$; $P(\lambda) = (-\lambda)^2 + 16 \Rightarrow \lambda = \pm 4i$.

$\leadsto$ Stable center

(although linearization is not conclusive this is consistent w/ fig 6.4.12)

6.4.15 $x'' + 4x - x^2 = 0 \iff \begin{cases} x' = y \\ y' = -4x + x^2 \end{cases}$

crit pts:

$\begin{cases} y = 0 \\ \text{and} \\ x(-4+x) = 0 \end{cases}$

crit pts are $(0,0)$ and $(4,0)$.

Linearization study:

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ -4+2x & 0 \end{bmatrix}$$

At $(0,0)$:

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}; \quad p(\lambda) = (-\lambda)^2 + 4 \Rightarrow \lambda = \pm 2i$$

⇝ consistent with stable center in fig. 6.4.13.

At $(4,0)$:

$$J(4,0) = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}; \quad p(\lambda) = (-\lambda)^2 - 4 \Rightarrow \lambda = \pm 2$$

⇝ unstable saddle point.

6.4.16 $x'' + 4x - 5x^3 + x^5 = 0 \iff \begin{cases} x' = y \\ y' = x(-4 + 5x^2 - x^4) \end{cases}$

crit pts satisfy:

$\begin{cases} y = 0 \\ \text{and} \\ x(-4 + 5x^2 - x^4) = 0 \end{cases}$

To solve $x^4 - 5x^2 + 4 = 0$ we let $u = x^2$ and solve $u^2 - 5u + 4 = 0$

$$u = \frac{5 \pm \sqrt{25-16}}{2} = 4 \text{ or } 1$$

$\Rightarrow x = \pm 2$ or $x = \pm 1$.

Thus critical points are $(0,2)$, $(0,-2)$, $(0,1)$, $(0,-1)$ and $(0,0)$.

Linearization study:

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ -4 + 15x^2 - 5x^4 & 0 \end{bmatrix}$$

At $(0,0)$: $J(0,0) = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$

eigenvalues are: $\lambda = \pm 2i$

$\Rightarrow$ stable center as in Fig 6.4.14

At $(0, \pm 2)$: $J(0,2) = J(0,-2) = \begin{bmatrix} 0 & 1 \\ -24 & 0 \end{bmatrix}$

eigenvalues are: $\lambda = \pm i\sqrt{24}$

$\Rightarrow$ Stable center as in Fig 6.4.14

At $(0, \pm 1)$: $J(0,1) = J(0,-1) = \begin{bmatrix} 0 & 1 \\ 6 & 0 \end{bmatrix}$

eigenvalues are $\lambda = \pm\sqrt{6}$

$\Rightarrow$ unstable saddle pt.