5.4.3 $x' = Ax$ where $A = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$

$$p(\lambda) = \det(A - \lambda I) = (1-\lambda)(5-\lambda) + 4 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$$

$\Rightarrow$ $\lambda = 3$ has (algebraic) multiplicity 2.

Eigenvector: Find $\underline{v}$ s.t. $(A - 3\lambda)\underline{v} = \underline{0}$

$$\begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Since we could find only one eigenvector we have to find a chain of generalized eigenvectors. Necessarily the chain must be of length 2.

$$(A - 3I)\underline{u}_1 = \underline{0}$$

$$(A - 3I)\underline{u}_2 = \underline{u}_1 \Rightarrow (A - 3I)^2\underline{u}_2 = (A - 3I)\underline{u}_1 = \underline{0}$$

So we look for $\underline{u}_2$ in $\ker(A - 3I)^2$:

$$(A - 3I)^2 = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

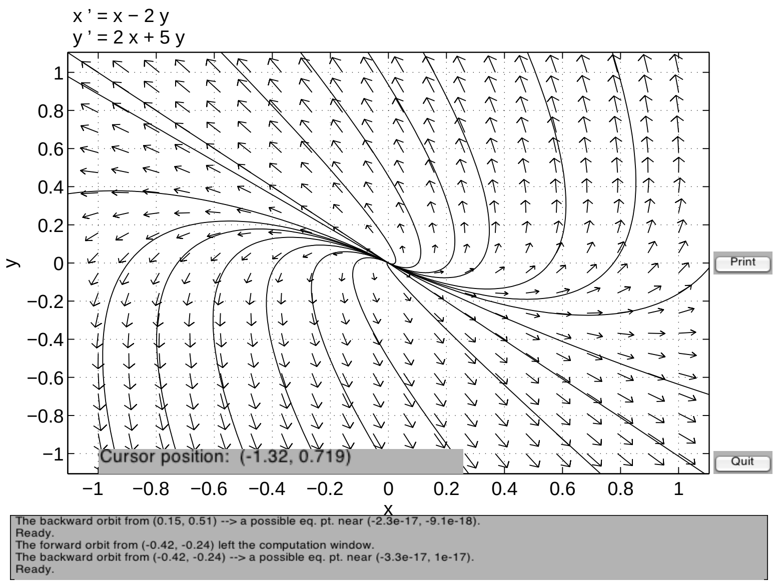
$\Rightarrow$ we can choose for $\underline{u}_2$ any vector in $\mathbb{R}^2$ (that's not an eigenvector).

For example: $\underline{u}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow (A - 3I)\underline{u}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2\underline{v}_1$ = eigenvector $= \underline{u}_1$

thus a general solution is of the form:

$$\boxed{\begin{aligned} \underline{x}(t) &= c_1 \underline{u}_1 e^{3t} + c_2 (t\underline{u}_1 + \underline{u}_2) e^{3t} \\ &= e^{3t}\left[c_1 \begin{bmatrix} -2 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2t + 1 \\ 2t \end{bmatrix}\right] \end{aligned}}$$

$$= e^{3t}\begin{bmatrix} -2c_1 + c_2 - 2c_2 t \\ 2c_1 + 2c_2 t \end{bmatrix}$$



(5.4, 10) $\quad \underline{x}' = A\,\underline{x}$ where $A = \begin{bmatrix} -13 & 40 & -48 \\ -8 & 23 & -24 \\ 0 & 0 & 3 \end{bmatrix}$

$p(\lambda) = \det(A - \lambda I) = -(\lambda-3)^2(\lambda-7) \Rightarrow \lambda = 3$ has algebraic mult. $= 2$.

Eigenvector:

$\lambda = 7$: $\quad (A - 7I)\underline{v} = \underline{0} \Leftrightarrow \begin{bmatrix} -20 & 40 & -48 \\ -8 & 16 & -24 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \underline{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = 3$: $\quad (A - 3I)\underline{v} = \underline{0} \Leftrightarrow \begin{bmatrix} -16 & 40 & -48 \\ -8 & 20 & -24 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ dim ker $A - 3I = 2 =$ geometric multiplicity

$\Rightarrow \lambda = 3$ is not defective: we can find two lin. indep eigenvectors

e.g. $\quad \underline{v}_2 = \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} \Rightarrow \boxed{\underline{x}(t) = c_1 e^{7t}\underline{v}_1 + c_2 e^{3t}\underline{v}_2 + c_3 e^{3t}\underline{v}_3}$

$\underline{v}_3 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

(5.4.12) $x' = Ax$ with $A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$

$p(\lambda) = \det(A - \lambda I) = -(\lambda+1)^3 \Rightarrow \lambda = -1$ has algebraic multiplicity 3.

Eigenvectors: find v s.t. $(A+I)v = 0$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

thus $\dim \ker(A+I) = 1$ = geometric multiplicity $\Rightarrow$ defect of eigenvalue is 2.
the only possibility here is to have a chain of length 3:

$$(A+I)u_1 = 0$$
$$(A+I)u_2 = u_1$$
$$(A+I)u_3 = u_2 \quad \Rightarrow \quad (A+I)^3 u_3 = 0$$

We can take for u_3 (almost) any vector in $\mathbb{R}^3$:

say $u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad (A+I)u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = u_2$

$$(A+I)u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = u_1 = v_1$$

Thus general sol is of the form:

$$x(t) = c_1 e^{-t} u_1 + c_2 e^{-t}(t u_1 + u_2) + c_3 e^{-t}\left(\frac{t^2}{2} u_1 + t u_2 + u_3\right)$$

$$= e^{-t} \begin{bmatrix} c_1 + c_3 + c_2 t + c_3 t^2/2 \\ c_1 + c_2 t + c_3 t^2/2 \\ c_2 + c_3 t \end{bmatrix}$$

5.5.2

$$\begin{cases} \underline{x}' = A\underline{x} \\ \underline{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \end{cases} \quad \text{with } A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$$

$$p(\lambda) = (2-\lambda)^2 - 4 = \lambda^2 - 4\lambda = \lambda(\lambda - 4)$$

eigenvectors:

$\lambda_1 = 0$:

$$\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\lambda_2 = 4$:

$$\begin{bmatrix} -2 & -1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$\Rightarrow$ two lin indep sol'ns: $\underline{x}_1(t) = e^{0t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

and $\underline{x}_2(t) = e^{4t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$\Rightarrow \quad \Phi(t) = \begin{bmatrix} 1 & e^{4t} \\ 2 & -2e^{4t} \end{bmatrix} \qquad \Phi(0) = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$$

$$\Phi(0)^{-1} = -\frac{1}{4}\begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} \underline{x}(t) &= \Phi(t)\,\Phi(0)^{-1}\underline{x}(0) \\ &= \Phi(t)\,\frac{1}{4}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \frac{3}{4}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{5}{4}\begin{bmatrix} e^{4t} \\ -2e^{4t} \end{bmatrix} \end{aligned}}$$

5.5.6

$$\begin{cases} \underline{x}' = A\underline{x} \\ \underline{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \end{cases} \qquad A = \begin{bmatrix} 7 & -5 \\ 4 & 3 \end{bmatrix}$$

$$p(\lambda) = (7-\lambda)(3-\lambda) + 20 = \lambda^2 - 10\lambda + 41$$

roots are $\lambda = 5 \pm 4i$. Since A has real entries we can focus on finding a (complex) eigenvector for $\lambda_1 = 5+4i$: $(A - \lambda_1 I)\underline{v} = \underline{0}$ $\ominus$

$$\begin{bmatrix} 2-4i & -5 \\ 4 & -2-4i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The two equations are the same since $\frac{2+4i}{5} \times$ first eq $=$ second eq.

5.5.6 (cont'd)

Thus an eigenvector for $\lambda_1 = 5+4i$ is $\underline{v}_1 = \begin{bmatrix} 5 \\ 2-4i \end{bmatrix}$

we have two linear independent solutions:

$$\mathrm{Re}\,(e^{\lambda_1 t}\,\underline{v}_1) = e^{5t}\left[\cos 4t \begin{pmatrix} 5 \\ 2 \end{pmatrix} - \sin 4t \begin{pmatrix} 0 \\ -4 \end{pmatrix}\right]$$

$$\mathrm{Im}\,(e^{\lambda_1 t}\,\underline{v}_1) = e^{5t}\left[\sin 4t \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \cos 4t \begin{pmatrix} 0 \\ -4 \end{pmatrix}\right]$$

$$\Rightarrow \Phi(t) = e^{5t}\begin{bmatrix} 5\cos 4t & 5\sin 4t \\ 2\cos 4t + 4\sin 4t & 2\sin 4t - 4\cos 4t \end{bmatrix}$$

$$\Phi(0) = \begin{bmatrix} 5 & 0 \\ 2 & -4 \end{bmatrix} \quad \text{and} \quad \Phi(0)^{-1} = -\frac{1}{20}\begin{bmatrix} -4 & 0 \\ -2 & 5 \end{bmatrix}$$

⚠ book sol has typo!

$$\Rightarrow \boxed{\underline{x}(t) = \Phi(t)\,\Phi(0)^{-1}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \Phi(t)\,\frac{1}{20}\begin{bmatrix} 8 \\ 4 \end{bmatrix} = e^{5t}\begin{bmatrix} 2\cos 4t + \sin 4t \\ 2\sin 4t \end{bmatrix}}$$

5.5.21 $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ is nilpotent since:

$$A^2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \boxed{e^{At} = I + tA = \begin{bmatrix} 1+t & -t \\ t & +1-t \end{bmatrix}}$$

5.5.27

$$\underline{x}' = \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}\underline{x} \; ; \; \underline{x}(0) = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$A = I + N$ where $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $N = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ is nilpotent since:

$$N^2 = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad N^3 = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

5.5.27 cont'd

Thus:
$$e^{(I+N)t} = e^{It}e^{Nt} = e^t\left(I + Nt + N^2\frac{t^2}{2}\right)$$
$$= e^t\begin{bmatrix} 1 & 2t & 3t+2t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{bmatrix}$$

5.5.31 Suppose $A, B \in \mathbb{R}^{n\times n}$ commute i.e. $AB = BA$ show
$e^{A+B} = e^A e^B$.

We have:
$$(A+B)^2 = A^2 + AB + BA + B^2 = A^2 + 2AB + B^2 \quad (AB = BA)$$
$$(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$$
$$\vdots$$
$$(A+B)^n = \sum_{k=0}^{n}\binom{n}{k}A^{n-k}B^k \quad \text{where} \quad \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

We can use the usual binomial formula here because A, B commute.

Now:
$$e^A e^B = \left(I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \cdots\right)\left(I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \frac{B^4}{4!} + \cdots\right)$$
$$= I + A + B + \underbrace{\frac{1}{2!}(A^2 + 2AB + B^2)}_{\text{all terms of degree 2}} + \underbrace{\frac{1}{3!}\left(A^3 + \frac{3!}{2!}A^2B + \frac{3!}{2!}AB^2 + B^3\right)}_{\text{all terms of degree 3}}$$
$$+ \frac{1}{4!}\left(A^4 + \frac{4!}{3!}A^3B + \frac{4!}{2!\,2!}A^2B^2 + \frac{4!}{3!}AB^3 + B^4\right)$$
$$+ \cdots$$
$$= I + (A+B) + \frac{1}{2!}(A+B)^2 + \frac{1}{3!}(A+B)^3 + \frac{1}{4!}(A+B)^4 + \cdots$$
$$= e^{A+B}. \quad \text{QED.}$$

5.5.36 $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ $p(\lambda) = -(\lambda-1)^3$

$\lambda = 1$ has multiplicity 3

eigenvectors: $(A - I)\underline{v} = \underline{0}$

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightsquigarrow \underline{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Since $\dim \ker(A - I) = 1$
$\lambda = 1$ is a defective eigenvalue of defect 2.

$\rightsquigarrow$ we need to find a chain s.t.

$$\begin{aligned} (A-I)\underline{u}_1 &= \underline{0} \\ (A-I)\underline{u}_2 &= \underline{u}_1 \\ (A-I)\underline{u}_3 &= \underline{u}_2 \end{aligned} \implies (A-I)^3 \underline{u}_3 = 0$$

since $(A-I)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ we can choose $\underline{u}_3$ to be almost any vector in $\mathbb{R}^3$.

Let us try: $\underline{u}_3 = \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$: $\implies (A - I)\underline{u}_3 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \underline{u}_2$

$$(A - I)\underline{u}_2 = \underline{0}$$

note: $\underline{e}_2$ was a bad choice to start (or end?) the chain!

Let us try $\underline{u}_3 = \underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \implies (A - \cancel{I})\underline{u}_3 = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} = \underline{u}_2$

$$\implies (A - I)\underline{u}_2 = \begin{pmatrix} 8 \\ 0 \\ 0 \end{pmatrix} = \underline{u}_1$$

$$(A - I)\underline{u}_1 = \underline{0}$$

$\implies \underline{\Phi}(t) = [\underline{x}_1(t) \quad \underline{x}_2(t) \quad \underline{x}_3(t)]$ where:

$$\begin{aligned} \underline{x}_1(t) &= e^t \underline{u}_1 \\ \underline{x}_2(t) &= e^t(\underline{u}_1 t + \underline{u}_2) \\ \underline{x}_3(t) &= e^t(\underline{u}_1 \frac{t^2}{2} + \underline{u}_2 t + \underline{u}_3) \end{aligned}$$

$$\underline{\Phi}(t) = e^t \begin{bmatrix} 8 & 8t+3 & 4t^2+3t \\ 0 & 4 & 4t \\ 0 & 0 & 1 \end{bmatrix} \qquad \underline{\Phi}(0) = \begin{bmatrix} 8 & 3 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

thus :

$$e^{At} = \Phi(t)\,\Phi(0)^{-1} \underset{\substack{\uparrow \\ \text{Maple}}}{=} \boxed{e^{t}\begin{bmatrix} 1 & 2t & 4t^2+3t \\ 0 & 1 & 4t \\ 0 & 0 & 1 \end{bmatrix}}$$