MATH 2280-2 HW6 Solution

5.2.2 $\underline{x}' = A\underline{x}$ where $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$

char poly $p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4$

$= (\lambda - 4)(\lambda + 1)$

eigenvalues are $\lambda_1 = 4$, $\lambda_2 = -1$.

eigenvectors:

- Find $\underline{v_1}$ s.t. $(A - 4I)\underline{v_1} = 0$: $\begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$

  take eg. $v_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

- find $\underline{v_2}$ st. $(A + I)\underline{v_2} = 0$: $\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$

  $\Rightarrow$ take e.g. $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Thus the general solution is $\boxed{\underline{x}(t) = c_1 e^{4t} \begin{pmatrix} 3 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

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> restart: with(DEtools): sys:=[diff(x(t),t) = 2*x(t)+3*y(t), diff
(y(t),t) = 2*x(t) + y(t)]:
DEplot(sys,[x(t),y(t)],t=-2..2,
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(0)=0]],
    x=-10..10,y=-10..10,linecolor=black);
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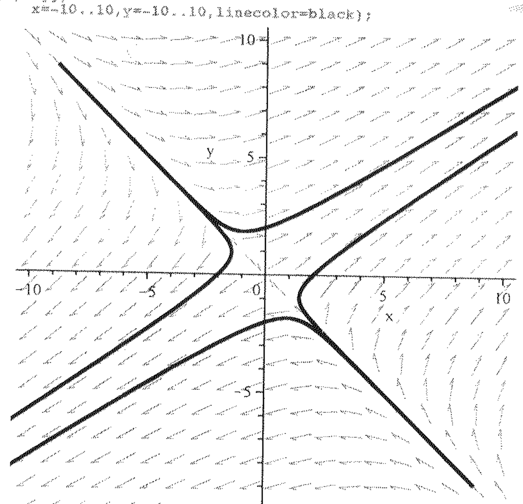

5.2.8 $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $x' = Ax$ with $A = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}$.

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = -(1-\lambda)(1+\lambda) + 5 = \lambda^2 + 4$$

two eigenvalues $\lambda_1 = 2i$ and $\lambda_2 = -2i$. (Conjugate pair)
We look for $v_1 \in \ker(A - 2iI)$:

$$\begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The two equations are equivalent since eq2 $\times$ $1-2i$ = eq1

$$(-1-2i)(1-2i) = -1 + 4(-1) - 2i + 2i = -5$$

$\Rightarrow v_1 = \begin{pmatrix} 1+2i \\ 1 \end{pmatrix}$. The corresponding imaginary sol is

$$x_1(t) = \underbrace{(\cos 2t + i \sin 2t)}_{e^{i2t}} \begin{pmatrix} 1+2i \\ 1 \end{pmatrix}$$

If we want a real valued solution it suffices to take linear comb. of $\operatorname{Re} x_1$ and $\operatorname{Im} x_1$.

$$\operatorname{Re} x_1 = \cos 2t \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin 2t \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\operatorname{Im} x_1 = \cos 2t \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus a general solution to DE is:

$$x(t) = d_1 \operatorname{Re} x_1 + d_2 \operatorname{Im} x_1 = \begin{bmatrix} (d_1 + 2d_2)\cos 2t + (-2d_1 + d_2)\sin 2t \\ d_1 \cos 2t + d_2 \sin 2t \end{bmatrix}$$

- for plot: see book.
- you can check this is equiv. to books solution.

$d_1 = c_1 - 2c_2$ (books constants)
$d_2 = 2c_1 + c_2$

5.2.12 $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\quad x' = Ax$, $A = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix}$

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -5 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 5$$

$$= \lambda^2 - 4\lambda + 8$$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16-32}}{2} = 2 \pm 2i$$

We have a complex conjugate pair of eigenvalues.
Since the matrix is real, we can first find complex eigenvector of one of them and use it to get the one for $\bar{\lambda}$.

$\underline{\lambda_1 = 2+2i}$: Find $\underline{v}_1$ s.t.

$$(A - (2+2i)I)\underline{v}_1 = 0$$

$$\begin{bmatrix} -1-2i & -5 \\ 1 & 1-2i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \iff a + (1-2i)b = 0$$

(since 1st eq $= (-1-2i) \times$ 2nd eq)

$$\Rightarrow \underline{v}_1 = \begin{pmatrix} 1-2i \\ -1 \end{pmatrix}$$

Let $\underline{x}_1(t) = e^{(2+2i)t}\,\underline{v}_1$, then general sol is $x(t) = d_1 \operatorname{Re} \underline{x}_1(t) + d_2 \operatorname{Im} \underline{x}_1(t)$

Here we have: $\underline{x}_1(t) = e^{2t}(\cos 2t + i \sin 2t)\begin{pmatrix} 1-2i \\ -1 \end{pmatrix}$

$$\Rightarrow \operatorname{Re} \underline{x}_1(t) = e^{2t}\left[\cos 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \sin 2t \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right]$$

$$\operatorname{Im} \underline{x}_1(t) = e^{2t}\left[\cos 2t \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$$

$$\Rightarrow x(t) = e^{2t}\begin{bmatrix} (d_1 - 2d_2)\cos 2t + (2d_1 + d_2)\sin 2t \\ (-d_1 \quad)\cos 2t \quad (-d_2)\cos 2t \end{bmatrix}$$

This sol is equivalent to the books:

$$\begin{cases} -d_1 = c_1 + 2c_2 \\ -d_2 = -2c_1 + c_2 \end{cases} \Rightarrow \begin{cases} d_1 - 2d_2 = -5c_1 \\ 2d_1 + d_2 = -5c_2 \end{cases}$$

(note: Multiplication by a (non-zero) scalar preserves eigenvectors. My particular choice of eigenvectors is different from the book's by some complex factor but the solutions are the same).

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> restart: with(DEtools): sys:=[diff(x(t),t) = x(t)-5*y(t), diff(y(t),t) = x(t) + 3*y(t)]:
DEplot(sys,[x(t),y(t)],t=-2..2,
    [[x(0)=0,y(0)=1],[x(0)=0,y(0)=-1],[x(0)=1,y(0)=0],[x(0)=-1,y(0)=0]],
    x=-5..5,y=-5..5,linecolor=black);
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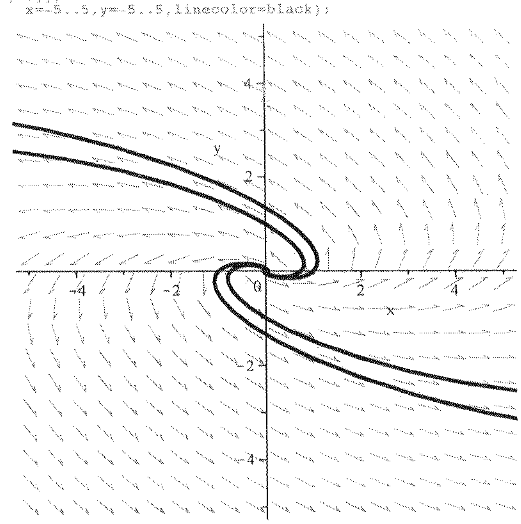
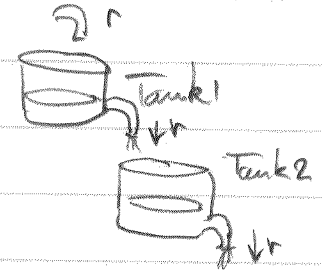


5.2.27



$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ = vector of quantity of salt in Tank 1 and 2.

DE satisfied by system: $\frac{d\underline{x}}{dt} = A\underline{x}$,

where $A = \begin{bmatrix} -k_1 & 0 \\ k_1 & -k_2 \end{bmatrix}$

and $k_1 = \frac{r}{V_1} = \frac{10}{50} = \frac{1}{5}$ $\Rightarrow A = \begin{bmatrix} -1/5 & 0 \\ 1/5 & -2/5 \end{bmatrix}$

$k_2 = \frac{r}{V_2} = \frac{10}{25} = \frac{2}{5}$

We use eigenvalue method to solve DE with $\underline{x}(0) = \begin{pmatrix} 15 \\ 0 \end{pmatrix}$.

$p(\lambda) = \det(A - \lambda I) = \begin{bmatrix} -1/5 - \lambda & 0 \\ 1/5 & -2/5 - \lambda \end{bmatrix} = (\lambda + \frac{1}{5})(\lambda + \frac{2}{5})$

$\lambda_1 = -\frac{1}{5}$, $\lambda_2 = -\frac{2}{5}$. Eigenvectors are: $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow \underline{x}(t) = c_1 e^{-t/5} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t/5} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\underline{x}(0) = \begin{pmatrix} c_1 \\ c_1 + c_2 \end{pmatrix}$ $\Rightarrow$ $c_1 = 15$, $c_2 = -15$

$$\Rightarrow \boxed{\underline{x}(t) = 15 e^{-t/5} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 15 e^{-2t/5} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.}$$

See book for plot

to find max amount of salt in 2nd tank: find t s.t.

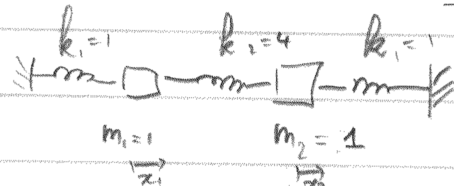
$$x_2'(t) = -\frac{15}{5} e^{-t/5} + \frac{2 \times 15}{5} e^{-2t/5} = 0$$

$$\Rightarrow -3e^{-t/5} + 6e^{-2t/5} = 0$$

$$\Rightarrow \quad e^{t/5} = 2 \quad \Rightarrow \quad t_{max} = 5 \ln 2$$

$$\boxed{\begin{aligned} x_2(t_{max}) &= 15\left(e^{-t_{max}/5} - e^{-2t_{max}/5}\right) \\ &= 15\left(\tfrac{1}{2} - \tfrac{1}{4}\right) = \tfrac{15}{4} = \boxed{3.75 \text{ pds.}} \end{aligned}}$$

5.3.2



$k = 1$

$M\underline{x}'' = K\underline{x}$ where $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $K = \begin{bmatrix} -5 & 4 \\ 4 & -5 \end{bmatrix}$

Thus we need to study eigenvalues and eigenvectors of $A = M^{-1}K = K$

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -5-\lambda & 4 \\ 4 & -5-\lambda \end{vmatrix} = (5+\lambda)^2 - 4 = (\lambda+1)(\lambda+9)$$

Eigenvalues $\lambda_1 = -1$, $\lambda_2 = -9$. Eigenvectors $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

the two natural (angular) freq. of system are $\omega_1 = \sqrt{-\lambda_1} = 1$

$\omega_2 = \sqrt{-\lambda_2} = 3$

- In the mode associated with ω_1, the 2 masses oscillate in phase, w/ same ampl.
- In the mode " " ω_2, " " " " out of phase w/ same ampl. (opposite directions)

5.3.8 We consider same system as above but with forcing $\underline{F} = \begin{bmatrix} 96 \cos t \\ 0 \end{bmatrix}$.

i.e.

$$M\underline{x}'' = K\underline{x} + \underline{F} \qquad (*)$$

We need to find a particular sol. $\underline{x}_p$ of $(*)$. We look for $\underline{x}_p$ of the form: $\underline{x}_p = \cos 3t \begin{pmatrix} a \\ b \end{pmatrix}$.

$$\underline{x}_p'' = -25\cos 5t \begin{pmatrix} a \\ b \end{pmatrix} \overset{(*)}{=} \begin{pmatrix} -5 & 4 \\ 4 & -5 \end{pmatrix} \cos(5t) \begin{pmatrix} a \\ b \end{pmatrix} + \cos(5t) \begin{pmatrix} 96 \\ 0 \end{pmatrix}$$

$\Rightarrow$ get lin syst. for $\begin{pmatrix} a \\ b \end{pmatrix}$:

$$\begin{pmatrix} 20 & 4 \\ 4 & 20 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -96 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \end{pmatrix} \Rightarrow \underline{x}_p = \cos 5t \begin{pmatrix} -5 \\ 1 \end{pmatrix}$$

Thus general sol. to $(*)$ is: $\underline{x}(t) = \underline{x}_p(t) + (a\cos t + b\sin t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$+ (c\cos 3t + d\sin 3t)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We find the constants to satisfy initial conditions:

$\underline{x}(0) = \underline{x}'(0) = \underline{0}$.

$$\underline{x}(0) = \underline{0} = \begin{pmatrix} -5 \\ 1 \end{pmatrix} + a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

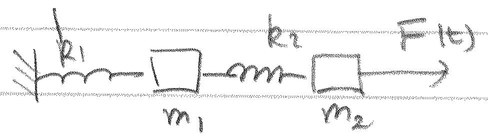
$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \Rightarrow \boxed{\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}}$$

$$\underline{x}'(0) = \underline{0} = b\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3d\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \underline{x}(t) = \cos 5t \begin{pmatrix} -5 \\ 1 \end{pmatrix} + 2\cos t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3\cos 3t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Supperposition of three oscillators.

5.3.15



$M\underline{x}'' = K\underline{x} + \underline{F}$

here $m_1 = 1$, $m_2 = 1/2$, $k_1 = 75$, $k_2 = 25$, $F(t) = 100\cos 10t$

so:

$$M = \begin{bmatrix} 1 & \\ & 1/2 \end{bmatrix}, \quad K = \begin{bmatrix} -(k_1+k_2) & k_2 \\ k_2 & -k_2 \end{bmatrix} = \begin{bmatrix} -100 & 25 \\ 25 & -25 \end{bmatrix}$$

We get second order system:

$$\underline{x}'' = \underbrace{\begin{bmatrix} -50 & 25/2 \\ 50 & -50 \end{bmatrix}}_{= M^{-1}K =: A} \underline{x} + \underbrace{\begin{bmatrix} 0 \\ 200 \end{bmatrix}}_{= M^{-1}\underline{F}} \cos 10t \qquad (*)$$

A particular sol to $(*)$ is of the form

$$\underline{x}_p = \begin{pmatrix} a \\ b \end{pmatrix} \cos 10t.$$

Plug in this $\underline{x}_p$ into $(*)$ we get system:

$$\begin{bmatrix} -50 & -25/2 \\ -50 & -50 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 200 \end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{bmatrix} 4/3 \\ -16/3 \end{bmatrix} \Rightarrow \underline{x}_p(t) = \cos 10t \begin{bmatrix} 4/3 \\ -16/3 \end{bmatrix}$$

Now the eigenvectors / values of A are:

$$\lambda_1 = -25, \quad \underline{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \omega_1 = 5$$

$$\lambda_2 = -75, \quad \underline{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \qquad \omega_2 = 5\sqrt{3}$$

Thus:

$$\underline{x}(t) = \cos 10t \begin{bmatrix} 4/3 \\ -16/3 \end{bmatrix} + (a\cos 5t + b\sin 5t)\begin{pmatrix} 1 \\ 2 \end{pmatrix} + (c\cos 5\sqrt{3}t + d\sin 5\sqrt{3}t)($$

We now find the constants a, b, c and d so that $\underline{x}(0) = \underline{x}'(0) = \underline{0}$

$$\underline{x}(0) = \underline{0} = \begin{bmatrix} 4/3 \\ -16/3 \end{bmatrix} + a\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} a \\ c \end{pmatrix} = \begin{bmatrix} -4/3 \\ 16/3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} a \\ c \end{pmatrix} = \begin{bmatrix} 2/3 \\ 2 \end{bmatrix}$$

$$\underline{x}'(0) = \underline{0} = 5b\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 5\sqrt{3}\,d\begin{pmatrix} -1 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \boxed{\underline{x}(t) = \cos 10t\begin{bmatrix} 4/3 \\ -16/3 \end{bmatrix} + \cos 5t\begin{bmatrix} 2/3 \\ 4/3 \end{bmatrix} + \cos 5\sqrt{3}t\begin{pmatrix} -2 \\ 4 \end{pmatrix}}$$

note book has a typo in solution.