HW 5 Solutions MATH 2280-2

3.5.49 $y'' - 4y' + 4y = 2e^{2x}$ to solve using variation of parameters

Char poly: $p(r) = r^2 - 4r + 4 = (r-2)^2$

(1) $\Rightarrow y_H(x) = a e^{2x} + b x e^{2x}$

(2) $y_H'(x) = a' e^{2x} + b' x e^{2x} + a(e^{2x})' + b(x e^{2x})'$

(2') set $a' e^{2x} + b' x e^{2x} = 0$.

(3) $y_H''(x) = a'(e^{2x})' + b'(x e^{2x})' + a(e^{2x})'' + b(x e^{2x})''$

$\leadsto$ We get a system of equations for a' and b'

(2)' : $a' e^{2x} + b' x e^{2x} = 0$

(3) $- 4\times(2) + 4\times(1)$: $a' 2e^{2x} + b'(1+2x)e^{2x} = 2e^{2x}$

$$\Leftrightarrow \quad e^{2x}\begin{bmatrix} 1 & x \\ 2 & 1+2x \end{bmatrix}\begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} 0 \\ 2e^{2x} \end{bmatrix}$$

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = \frac{e^{2x}}{e^{4x}(1+2x-2x)}\begin{bmatrix} 1+2x & -x \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2e^{2x} \end{bmatrix}$$

$$= e^{-2x}(2e^{2x})\begin{bmatrix} -x \\ 1 \end{bmatrix} = 2\begin{pmatrix} -x \\ 1 \end{pmatrix}$$

$\Rightarrow a(x) = \int -2x\,dx = -x^2$

$b(x) = \int 2\,dx = 2x$

$\Rightarrow$ a particular sol is:

$y_P(x) = -x^2 e^{2x} + 2x^2 e^{2x} = x^2 e^{2x}$.

Since we did a problem similar to 3.5.49 in class, here is another example (<u>not</u> in HW) of variation of parameters

(3.5.52) $y'' + 9y = \sin 3x$ to solve using var of params.

$p(r) = r^2 + 9 = (r+3i)(r-3i)$

(1) $y_H(x) = a\cos 3x + b\sin 3x$

(2) $y_H'(x) = a'\cos 3x + b'\sin 3x + a(\cos 3x)' + b(\sin 3x)'$

(2') we set $a'\cos 3x + b'\sin 3x = 0$

(3) $y_H''(x) = a'(\cos 3x)' + b'(\sin 3x)' + a(\cos 3x)'' + b(\sin 3x)''$

We get two eq for a' and b':

$$\begin{cases} a'\cos 3x + b'\sin 3x = 0 & (2') \\ -3a'\sin 3x + 3b'\cos 3x = \sin 3x & (3) + 9\times(1) \end{cases}$$

$$\Leftrightarrow \begin{bmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{bmatrix}\begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} 0 \\ \sin 3x \end{bmatrix}$$

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = \frac{1}{3\cos^2 3x + 3\sin^2 3x}\begin{bmatrix} 3\cos 3x & -\sin 3x \\ 3\sin 3x & \cos 3x \end{bmatrix}\begin{bmatrix} 0 \\ \sin 3x \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -\sin^2 3x \\ \sin 3x\cos 3x \end{bmatrix}$$

$$\Rightarrow a(x) = \int -\frac{\sin^2 3x}{3}\,dx = \frac{1}{3}\left(\frac{1}{6}\cos 3x\sin 3x - \frac{1}{2}x\right)$$

$$b(x) = \int \frac{1}{3}\sin 3x\cos 3x\,dx = -\frac{1}{18}\cos^2 3x$$

$$\Rightarrow y_p(x) = \frac{1}{18}\cos^2 3x\sin 3x - \frac{x}{6}\cos 3x - \frac{1}{18}\cos^2 3x\sin 3x$$

$$= \boxed{-\frac{x}{6}\cos 3x}$$

3.6.2 $x'' + 4x = 5\sin 3t \qquad x(0) = x'(0) = 0$

$x_H(t) = a\cos 2t + b\sin 2t$

$x_p(t) = c\sin 3t \Rightarrow \quad -9c\sin 3t + 4c\sin 3t = 5\sin 3t$

$\Rightarrow \quad c = -1$

$\Rightarrow x_p(t) = -\sin 3t$

$\Rightarrow x(t) = x_H(t) + x_p(t) = a\cos 2t + b\sin 2t - \sin 3t$

$x(0) = 0 = a$

$x'(0) = 2b - 3 = 0 \quad \Rightarrow \quad b = \frac{3}{2}$

$\Rightarrow x(t) = \frac{3}{2}\sin 2t - \sin 3t$, period is 2π

plot: see back

3.6.11 $L(x) = x'' + 4x' + 5x = 10\cos 3t \qquad x(0) = x'(0) = 0$

Look for x_p in $V = \text{span}\{\cos 3t, \sin 3t\}$.

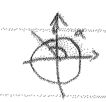
We have: $L(\cos 3t) = -9\cos 3t - 12\sin 3t + 5\cos 3t = -4\cos 3t - 12\sin 3t$

$L(\sin 3t) = -9\sin 3t + 12\cos 3t + 5\sin 3t = 12\cos 3t - 4\sin 3t$

thus: $(L)_{\mathcal{B}} = \begin{bmatrix} -4 & 12 \\ -12 & -4 \end{bmatrix}$ and if $x_p(t) = a\cos 3t + b\sin 3t$:

$(10\cos 3t)_{\mathcal{B}} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} \qquad \begin{bmatrix} -4 & 12 \\ -12 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$



$$\Rightarrow x_p(t) = -\frac{1}{4}\cos 3t + \frac{3}{4}\sin 3t = \frac{\sqrt{10}}{4}\left(-\frac{1}{\sqrt{10}}\cos 3t + \frac{3}{\sqrt{10}}\sin 3t\right)$$

$$= \frac{\sqrt{10}}{4}\cos(3t - \alpha)$$

where $\alpha = \operatorname{acos}\left(-\frac{1}{\sqrt{10}}\right) \simeq 1.8925$

Characteristic polynomial is $p(r) = r^2 + 4r + 5$ roots are $r = -2 \pm i$

$\Rightarrow x_h(t) = e^{-2t}(a\cos t + b\sin t)$

$\Rightarrow x(t) = \frac{\sqrt{10}}{4}\cos(3t - \alpha) + \underbrace{e^{-2t}(a\cos t + b\sin t)}_{x_p(t)}$

Using init. cond:

$$x(0) = \frac{\sqrt{10}}{4}\cos(\alpha) + a = 0$$

$$\Rightarrow a = \left(-\frac{\sqrt{10}}{4}\right)\left(-\frac{1}{\sqrt{10}}\right) = \frac{1}{4}$$

$$x'(t) = -\frac{3}{4}\sqrt{10}\sin(3t - \alpha) - e^{-2t}\left((2a - b)\cos t + (a + 2b)\sin t\right)$$

$$x'(0) = +\frac{3}{4}\sqrt{10}\underbrace{\sin\alpha}_{3/\sqrt{10}} - (2a - b) = \frac{9}{4} - \frac{1}{2} + b = 0$$

$\Rightarrow b = -7/4$

$$\Rightarrow x_p(t) = e^{-2t}\left[\frac{1}{4}\cos t - \frac{7}{4}\sin t\right] = \frac{\sqrt{50}}{4}e^{-2t}\left[\frac{1}{\sqrt{50}}\cos t - \frac{7}{\sqrt{50}}\sin t\right]$$

$$= \frac{\sqrt{50}}{4}e^{-2t}\cos(t - \beta) \qquad \beta = 2\pi + \operatorname{asin}\left(-\frac{7}{\sqrt{50}}\right) \approx 4.8543$$

plot: see book

3.6.16 $L(x) = mx'' + cx' + kx = F_0 \cos\omega t$ (mass-spring system)

where $m=1,\ c=4,\ k=5,\ F_0=2$.

note: char. poly. of DE is same as in 3.6.11 $\Rightarrow$ roots $r = -2 \pm i$

we only need to find particular solution to DE. $\Rightarrow$ use undetermined coeff method

look for $x_p \in V = \text{span}\{\cos\omega t, \sin\omega t\}$

$$L(\cos\omega t) = -\omega^2\cos\omega t - 4\omega\sin\omega t + 5\cos\omega t = (5-\omega^2)\cos\omega t - 4\omega\sin\omega t$$

$$L(\sin\omega t) = -\omega^2\sin\omega t + 4\omega\cos\omega t + 5\sin\omega t = 4\omega\cos\omega t + (5-\omega^2)\sin\omega t$$

Thus the matrix A of L restricted to V is:

$$A = \begin{bmatrix} 5-\omega^2 & -4\omega \\ 4\omega & 5-\omega^2 \end{bmatrix}$$

Thus if $x_p = a\cos\omega t + b\sin\omega t$: $\begin{bmatrix} a \\ b \end{bmatrix} = A^{-1}\begin{bmatrix} 10 \\ 0 \end{bmatrix}$

$$= \frac{10}{(5-\omega^2)^2 + 16\omega^2}\begin{bmatrix} 5-\omega^2 \\ 4\omega \end{bmatrix}$$

The amplitude of the steady periodic solution is:

$$C(\omega) = \sqrt{a^2+b^2} = \left(\frac{10^2}{[(5-\omega^2)^2+16\omega^2]^2}\left((5-\omega^2)^2 + 16\omega^2\right)\right)^{1/2}$$

$$= \frac{10}{\sqrt{(5-\omega^2)^2+16\omega^2}} = \frac{10}{\sqrt{25+6\omega^2+\omega^4}}$$

$$C'(\omega) = -\frac{5(4t^3+12t)}{(25+6\omega^2+\omega^4)^{3/2}} \leq 0$$ thus $C(\omega)$ is monotonically decreasing and there is no practical resonance.

For plot of $C(\omega)$ see book.

4.1.2 $x^{(4)} + 6x'' - 3x' + x = \cos 3t$ in the form of a first order system

$x = x_0$

$x_0' = x_1$

$x_1' = x_2$

$x_2' = x_3$

$x_3' = -6x_2 + 3x_1 - x_0 + \cos 3t$

ie: letting $\underline{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\Leftrightarrow \dfrac{d\underline{x}}{dt} = A\,\underline{x} + \underline{F}$

where

$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 3 & -6 & 0 \end{bmatrix}$

$\underline{F} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cos 3t \end{bmatrix}$

4.1.3 $t^2 x'' + t x' + (t^2 - 1)x = 0$

$x = x_0$

$x_0' = x_1$

$t^2 x_1' = -t x_1 + (t^2 - 1) x_0$

4.1.13

$\begin{cases} x' = -2y \\ y' = 2x \\ x(0) = 1,\ y(0) = 0 \end{cases}$ $\Rightarrow y = -\frac{1}{2}x'$

$\Rightarrow -\frac{1}{2}x'' = 2x$

$x'' + 4x = 0$

$\Rightarrow x(t) = a\cos 2t + b\sin 2t.$

$x(0) = a = 1$

$y(0) = -\frac{1}{2}x'(0) = -\frac{1}{2}(2b) = 0$

thus $\boxed{\begin{aligned} x(t) &= \cos 2t \\ y(t) &= \sin 2t \end{aligned}}$ See book for plot

4.1.16

$$\begin{cases} x' = 8y \\ y' = -2x \end{cases} \Rightarrow y = \frac{x'}{8}$$

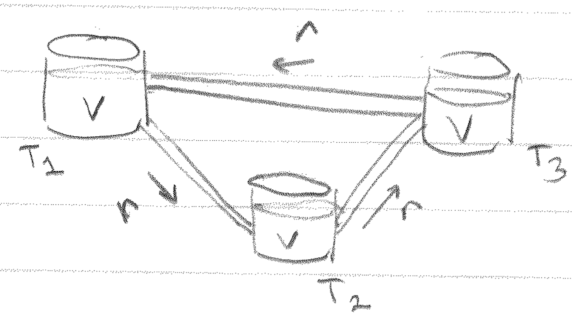
$$\Rightarrow \frac{x''}{8} = -2x$$

$$\Rightarrow x'' + 16x = 0$$

$$\Rightarrow \boxed{\begin{aligned} x(t) &= a\cos 4t + b\sin 4t \\ y(t) &= \frac{x'(t)}{8} = -\frac{a}{2}\sin 4t + \frac{b}{2}\cos 4t \end{aligned}}$$

See book for plot.

4.1.26



In tank 1:

concentration in tank 3

(" in tank 1

$$\frac{dx_1}{dt} = r\boxed{\frac{x_3}{V}} - r\boxed{\frac{x_1}{V}}$$

$$= \frac{10}{100}(x_3 - x_1)$$

Similarly:

$$\Rightarrow \begin{cases} 10\,x_1' = x_3 - x_1 \\ 10\,x_2' = x_1 - x_2 \\ 10\,x_3' = x_2 - x_3 \end{cases}$$

5.1.13

$$\begin{cases} x' = 2x + 4y + 3e^t \\ y' = 5x - y - t^2 \end{cases}$$

Let $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ then system is $\underline{x}' = A\underline{x} + \underline{F}(t)$

where $A = \begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix}$ and $\underline{F}(t) = \begin{bmatrix} 3e^t \\ -t^2 \end{bmatrix}$

5.1.18

$$\begin{cases} x' = tx - y + e^t z \\ y' = 2x + t^2 y - z \\ z' = e^{-t}x + 3ty + t^3 z \end{cases}$$

Let $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then system is $\underline{x}' = A(t)\,\underline{x}$

with $A = \begin{bmatrix} t & -1 & e^t \\ 2 & t^2 & -1 \\ e^{-t} & 3t & t^3 \end{bmatrix}$

5.1.22 $\underline{x}' = \underbrace{\begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}}_{A} \underline{x}$. We have $\underline{x}_1 = e^{3t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \underline{x}_1' = 3e^{3t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Moreover $A\underline{x}_1 = e^{3t}\begin{pmatrix} -3 & 2 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$= e^{3t}\begin{pmatrix} 3 \\ 9 \end{pmatrix} \Rightarrow \underline{x}_1$ solves $\underline{x}_1' = A\underline{x}_1$.

For $\underline{x}_2 = e^{-2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\Rightarrow \underline{x}_2' = -2e^{-2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Moreover $A\underline{x}_2 = e^{-2t}\begin{pmatrix} -3 & 2 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = e^{-2t}\begin{bmatrix} -4 \\ -2 \end{bmatrix} \Rightarrow \underline{x}_2' = A\underline{x}_2$.

The two solutions are lin. indep. since:

$W(t) = \left|\underline{x}_1 \ \underline{x}_2\right| = \begin{vmatrix} e^{3t} & 2e^{-2t} \\ 3e^{3t} & e^{-2t} \end{vmatrix} = e^{t}(1-6) = -5e^{t} \neq 0 \ \forall t.$

A general sol to system is of the form $\underline{x} = a\underline{x}_1 + b\underline{x}_2$.

5.1.27 $\underline{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\underline{x}$. $\underline{x}_1 = e^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \underline{x}_1' = 2e^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$A\underline{x}_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}e^{2t} \Rightarrow \underline{x}_1' = A\underline{x}_1$.

$\underline{x}_2 = e^{-t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow \underline{x}_2' = e^{-t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$; $A\underline{x}_2 = e^{-t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \underline{x}_2' = A\underline{x}_2$.

$\underline{x}_3 = e^{-t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \underline{x}_3' = e^{-t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$; $A\underline{x}_3 = e^{-t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \underline{x}_3' = A\underline{x}_3$

The Wronskian is $W(t) = \left|\underline{x}_1 \ \underline{x}_2 \ \underline{x}_3\right| = e^{2t} \times e^{-t} \times e^{-t}\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{vmatrix}$

$= 2 + 1 = 3 \neq 0 \ \forall t$, sols are lin. indep.

A general sol to system is $\underline{x} = a\underline{x}_1 + b\underline{x}_2 + c\underline{x}_3$.

5.1.31 We must find solution with initial cond $\underline{x}_0 = \begin{pmatrix} 0 \\ 5 \end{pmatrix} = \underline{x}(0)$.

Since $\underline{x}(t) = a e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + b e^{-2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$= \begin{bmatrix} e^{3t} & 2e^{-2t} \\ 3e^{3t} & e^{-2t} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\Rightarrow \quad \underline{x}(0) = \begin{pmatrix} 0 \\ 5 \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \Rightarrow \boxed{\underline{x}(t) = 2 e^{t} \begin{pmatrix} 2 \\ -3 \end{pmatrix} - e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

5.1.36 We must find a solution with init cond $\underline{x}_0 = \underline{x}(0) = \begin{bmatrix} 10 \\ 12 \\ -1 \end{bmatrix}$

Since $\underline{x}(t) = a e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b e^{-t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}$

$$\underline{x}(0) = \begin{pmatrix} 10 \\ 12 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 5 \end{pmatrix} \Rightarrow \boxed{\underline{x}(t) = 7e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 3 e^{-t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + 5 e^{-t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}.$$