MATH 2280 HW 4 Solutions

.4

$$\begin{cases} y'' + 25y = 0 \\ y(0) = 10 \\ y'(0) = -10 \end{cases}, \quad y_1 = \cos 5x, \; y_2 = \sin 5x$$

We have $y_1'' = -25\cos 5x = -25y_1$

$y_2'' = -25\sin 5x = -25y_2$

The particular solution is of the form:

$$y = c_1 \cos 5x + c_2 \sin 5x$$

$$10 = y(0) = c_1 \cos(0) + c_2 \sin(0) = c_1$$

$$-10 = y'(0) = -5c_1 \sin(0) + 5c_2 \cos(0) = 5c_2$$

$$\Rightarrow c_1 = 10, \quad c_2 = -2$$

$$\Rightarrow \boxed{y = 10\cos 5x - 2\sin 5x}$$

3.1.5

optional)

$$\begin{cases} y'' - 3y' + 2y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}, \quad y_1 = e^x, \; y_2 = e^{2x}$$

we have

$y_1'' = e^x \quad \times 1$

$y_1' = e^x \quad \times -3$

$y_1 = e^x \quad \times 2$

$= 0.$

$y_2'' = 4e^{2x} \quad \times 1$

$y_1' = 2e^{2x} \quad \times (-$

$y_1 = e^{2x} \quad \times$

$= 0.$

Particular solution is of the form:

$$y = c_1 e^x + c_2 e^{2x}$$

$$1 = y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2$$

$$0 = y'(0) = c_1 e^0 + 2c_2 e^0 = c_1 + 2c_2$$

$$\Rightarrow c_1 = 2, \quad c_2 = -1$$

$$\boxed{\Rightarrow y = 2e^x - e^{2x}}$$

19

Consider the DE $yy'' + (y')^2 = 0$

$y_1 = 1$ is a solution since $y_1' = y_1'' = 0$.

$y_2 = \sqrt{x}$ is a solution since $y_2' = \frac{1}{2} x^{-\frac{1}{2}}$, $y_2'' = -\frac{1}{4} x^{-3/2}$

and: $y_2 y_2'' = x^{\frac{1}{2}} \left(-\frac{1}{4} x^{-3/2}\right) = -\frac{1}{4} x^{-1}$

$(y_2')^2 = \frac{1}{4} x^{-1}$

$\Rightarrow \quad y_2 y_2'' + (y_2')^2 = 0$

However: $y_1 + y_2 = 1+\sqrt{x}$ is not a solution to DE since:

$$(1+\sqrt{x})(1+\sqrt{x})'' + ((1+\sqrt{x})')^2$$
$$= (1+\sqrt{x})\left(-\frac{1}{4} x^{-3/2}\right) + \left(\frac{1}{2} x^{-\frac{1}{2}}\right)^2$$
$$= -\frac{1}{4} x^{-3/2} - \frac{1}{4} x^{-1} + \frac{1}{4} x^{-1} = -\frac{1}{4} x^{-3/2} \neq 0.$$

This is because DE is non-linear so the sum of solutions is not necessarily a solution.

3.1.29

Consider DE $x^2 y'' - 4xy' + 6y = 0$

Is $y_1 = x^2$ a solution?

$x^2 (x^2)'' - 4x(x^2)' + 6x^2 = 2x^2 - 8x^2 + 6x^2 = 0$ ✓ yes

Is $y_2 = x^3$ a solution?

$x^2 (x^3)'' - 4x(x^3)' + 6x^3 = 6x^3 - 12x^3 + 6x^3 = 0$ ✓ yes

We obviously have $y_1(0) = y_1'(0) = 0$ and $y_2(0) = y_2'(0) = 0$

So we have two distinct solutions with same initial data.

This does not contradict Theorem 2 because the DE in the form required for theorem 2 is: $y'' - \frac{4}{x} y' + \frac{6}{x^2} y = 0$, and neither $\frac{4}{x}$ or $\frac{6}{x^2}$ are cont. at

3.1.31 (optional)

The functions $y_1 = \cos x^2$ and $y_2 = \sin x^2$ are linearly independent because their quotient $\frac{y_2}{y_1} = \tan x^2$ is not a constant function.

Their Wronskian determinant is:

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x^2 & \sin x^2 \\ -2x\sin x^2 & 2x\cos x^2 \end{vmatrix} = 2x\left(\cos^2 x^2 + \sin^2 x^2\right) = 2x$$

and vanishes at $x = 0$.

If y_1 and y_2 were solutions of a DE

$$y'' + py' + qy = 0 \text{ with } p \text{ and } q \text{ cont.}$$

then by theorem 3 there are only two possibilities:

i) y_1 and y_2 are lin dep and $W(x) = 0 \ \forall x$
(this is obviously not the case since y_1 and y_2 are lin indep)

ii) y_1 and y_2 are lin indep and $W(x) \neq 0 \ \forall x$

Since $W(0) = 0$, ii) fails and thus y_1, y_2 cannot be solutions to $y'' + py' + qy = 0$, with p and q cont.

3.1.38

$$4y'' + 8y' + 3y = 0$$

Characteristic polynomial is $p(r) = 4r^2 + 8r + 3$
$= (r + \frac{1}{2})(r + \frac{3}{2})$

$\Rightarrow y_H = c_1 y_1 + c_2 y_2$ where $y_1 = e^{-\frac{1}{2}x}$

$y_2 = e^{-\frac{3}{2}x}$

3.1.43 Find a linear homog DE w/ constant coeff s.t.

$y(x) = c_1 + c_2 e^{-10x}$ is its general solution

The roots of char poly are 0 and -10

$\Rightarrow p(r) = r(r+10) = r^2 + 10r$

$\Rightarrow \boxed{y'' + 10y' = 0}$

3.2.8 Prove $f(x) = e^x$, $g(x) = e^{2x}$, $h(x) = e^{3x}$ are lin indep on $\mathbb{R}$.

We look at Wronskian:

$$W(x) = \begin{bmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{bmatrix} = \begin{bmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{bmatrix}$$

$$= e^x e^{2x} e^{3x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x}\left(\begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \right)$$

$$= 2e^{6x} \neq 0 \text{ on } \mathbb{R}$$

by multilinearity of determinant, ie. property that: $\det(\alpha \underline{a}, \beta \underline{b}, \gamma \underline{c}) = \alpha\beta\gamma \det(\underline{a}, \underline{b}, \underline{c})$

$\Rightarrow f, g, \& h$ are lin. indep on $\mathbb{R}$

3.2.13
$$\begin{cases} y^{(3)} + 2y'' - y' - 2y = 0 \\ y(0) = 1 \\ y'(0) = 2 \\ y''(0) = 0 \end{cases}, \quad y_1 = e^x, y_2 = e^{-x}, y_3 = e^{-2x}$$

A general sol to DE has form $c_1 y_1 + c_2 y_2 + c_3 y_3$

Thus the particular solution must satisfy:

$$1 = y(0) = c_1 e^0 + c_2 e^0 + c_3 e^0 = c_1 + c_2 + c_3$$
$$2 = y'(0) = c_1 e^0 - c_2 e^0 - 2c_3 e^0 = c_1 - c_2 - 2c_3$$
$$0 = y''(0) = c_1 e^0 + c_2 e^0 + 4c_3 e^0 = c_1 + c_2 + 4c_3$$

2.13 Cont'd

This equivalent to solving linear system:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \Rightarrow c_1 = \frac{4}{3},\ c_2 = 0,\ c_3 = -\frac{1}{3}.$$

$$\Rightarrow y(x) = \frac{1}{3}\left(4e^{x} - e^{-2x}\right)$$

3.2.21

$$\begin{cases} y'' + y = 3x \\ y(0) = 2 \\ y'(0) = -2 \end{cases} \qquad \begin{aligned} y_H &= c_1 \cos x + c_2 \sin x \\ y_p &= 3x \end{aligned}$$

Any sol to DE has form

$$y = y_p + y_H = 3x + c_1 \cos x + c_2 \sin x$$

$$\Rightarrow \quad 2 = y(0) = 3\times 0 + c_1 \cos(0) + c_2 \sin(0) = c_1$$

$$-2 = y'(0) = 3 - c_1 \sin(0) + c_2 \cos(0) = 3 + c_2$$

$$\Rightarrow c_1 = 2, \quad c_2 = -5$$

$$\Rightarrow \quad y = 3x + 2\cos x - 5 \sin x$$

3.3.9

$y'' + 8y' + 25y = 0$, has characteristic poly:

$$p(r) = r^2 + 8r + 25$$

$$p(r) = 0 \Rightarrow r = \frac{-8 \pm \sqrt{64 - 100}}{2}$$

$$= \frac{-8 \pm i6}{2} = -4 \pm 3i$$

Thus $y_H = c_1 y_1 + c_2 y_2$ where $y_1 = e^{(-4+3i)x}$, $y_2 e^{(-4+3i)x}$

or $= d_1 u_1 + d_2 u_2$ where $u_1 = e^{-4x}\cos 3x$, $u_2 = e^{-4x}\sin 3x$

3.3.10

$5y^{(4)} + 3y^{(3)} = 0$. Char poly is $p(r) = 5r^4 + 3r^3$
$= r^3(5r+3)$

$\Rightarrow$ $r = 0$ is root of multip. 3
$r = -\frac{3}{5}$ ——— 1

$$\Rightarrow y_H = \underbrace{c_1 + c_2 x + c_3 x^2}_{\text{corresp. to root } r=0} + \overbrace{c_4 e^{-\frac{3}{5}x}}$$

3.3.33

$y^{(3)} + 3y'' - 54y = 0$, $y = e^{3x}$ is a sol.
char poly is $p(r) = r^3 + 3r^2 - 54$.
Since e^{3x} is a sol, $r = 3$ is a root of char polyn. (check!)
Thus $p(r) = (r-3)\, q(r)$, where $q \in P_2$.
We find q by division of polynomials (Euclid's algo)

$$\begin{array}{l|l} r^3 + 3r^2 - 54 & r-3 \\ \hline \underline{r^3 - 3r^2} & r^2 + 6r + 18 = q(r) \\ \quad 6r^2 - 54 & \\ \quad \underline{6r^2 - 18r} & \\ \quad 18r - 54 & \\ \quad \underline{18r - 54} & \\ \qquad 0 & \end{array}$$

$q(r)$ has roots: $r = \dfrac{-6 \pm \sqrt{36 - 72}}{2} = -3 \pm 3i$

These roots correspond to solutions $\{e^{(-3+3i)x}, e^{(-3-3i)x}\}$
The subspace spanned by these two solutions is the same as that spanned by:

$$\{e^{-3x}\cos 3x, \ e^{-3x}\sin 3x\}$$

$\Rightarrow y_H = c_1 e^{3x} + c_2 e^{-3x}\cos 3x + c_3 e^{-3x}\sin 3x$

3.3.39 Find a linear homog DE with const. coeff that has general solution of the form:

$$y(x) = (A + Bx + Cx^2)e^{2x} \qquad (1)$$

If $L(y)$ = linear diff op of this DE, then

$$(1) \Leftrightarrow \ker L = \text{span}\{e^{2x}, xe^{2x}, x^2 e^{2x}\}$$

$\Rightarrow$ 2 is a root of char polyn. w/ multiplicity 3. (and it is only root)

$\Rightarrow$ $p(r) = (r-2)^3 = r^3 - 6r^2 + 12r - 8$

$$\boxed{\begin{array}{cccc} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{array}}$$

$\Rightarrow$ $L(y) = y^{(3)} - 6y'' + 12y' - 8y = 0.$

note: Pascal triangle = easy way to remember $(a+b)^n = \ldots$

3.4.5 The DE satisfied by pendulum 1 is:

$$L_1\theta'' + \frac{GM}{R_1^2}\theta = 0 \Rightarrow \text{natural freq } \omega_1 = \sqrt{\frac{GM}{L_1R_1^2}}$$

similarly for pendulum 2:

$$L_2\theta'' + \frac{GM}{R_2^2}\theta = 0 \Rightarrow \quad '' \quad \omega_2 = \sqrt{\frac{GM}{L_2R_2^2}}$$

the period of a pendulum is $p = \frac{2\pi}{\omega}$

$$\Rightarrow \boxed{\frac{p_1}{p_2} = \frac{2\pi/\omega_1}{2\pi/\omega_2} = \frac{\omega_2}{\omega_1} = \frac{\sqrt{L_1R_1^2}}{\sqrt{L_2R_2^2}} = \frac{R_1}{R_2}\sqrt{\frac{L_1}{L_2}}}$$

3.4.6 Let N be the number of pendulum swings in Paris / day. then the period of this pendulum is:

$$p_1 = \frac{24}{N} \text{ hours} = \frac{86400}{N} \text{ seconds}$$

At the equator, for the same N swings the pendulum takes 1 day + 2 min 40 s = 86400 + 160 s = 86560 s thus the period at the equator should be:

$$p_2 = \frac{86560}{N} \text{ seconds}$$

$\Rightarrow$ using previous problem:

$$\frac{p_1}{p_2} = \frac{R_1}{R_2}\cancel{\sqrt{\frac{L_1}{L_2}}}$$

1 since pendulum is the same.

where R_1 = rad. in Paris
R_2 = rad at equator

$$\Rightarrow \quad R_2 = R_1 \frac{p_2}{p_1}$$

$\Rightarrow$ equatorial bulge is $R_2 - R_1 = R_1\left(1 - \frac{p_2}{p_1}\right)$

$$= R_1 \frac{160}{86400} \simeq 7.33\text{km}$$

3.4.13 (optional) The DE satisfied by system is:

$$mx'' + cx' + kx = 0 \quad \text{with } m = 10 \quad c = 9 \quad , k = 2$$

(a) char poly of DE is: $p(r) = mr^2 + cr + k$ roots: $r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$

.4.13 cont'd

here:

$$c^2 - 4mk = 81 - 4.2.10 = 1 > 0$$

$\Rightarrow$ we are in the overdamped case and $r = \dfrac{-9 \pm 1}{20} = \begin{cases} -1/2 \\ \text{or} \\ -2/5 \end{cases}$

$$\Rightarrow \quad x(t) = Ae^{-1/2t} + Be^{-2/5t}$$

Using initial condition:

$$0 = x(0) = A + B$$
$$5 = x'(0) = -\frac{1}{2}A - \frac{2}{5}B$$

$$\Rightarrow 5 = -\frac{1}{2}A + \frac{2}{5}A = \frac{4-5}{10}A = -\frac{1}{10}A$$

$$\Rightarrow \quad A = -50 = -B$$

$$\Rightarrow \quad \boxed{x(t) = 50(e^{-2/5t} - e^{-1/2t})}$$

which precisely decays as $t \to \infty$.

(b) To find how far the mass moves we need to find T s.t.

$x'(T) = 0$ (at the farthest pt velocity is zero)

$$x'(T) = 50\left(-\frac{2}{5}e^{-2/5T} + \frac{1}{2}e^{-1/2T}\right) = 0$$

$$\Rightarrow \quad \frac{1}{2}e^{-1/2T} = +\frac{2}{5}e^{-2/5T}$$

$$\Rightarrow \quad e^{-1/10T} = \frac{4}{5}$$

$$\Rightarrow \quad T = -10\ln\left(\frac{4}{5}\right) = 10\ln(5/4)$$

$$\Rightarrow \boxed{x(T) = 50\left(\left(\frac{5}{4}\right)^{-4} - \left(\frac{5}{4}\right)^{-5}\right) = 4.096}$$

.4.14 (a) Same system as in previous exercise:

$$mx'' + cx' + kx = 0 \quad \text{with } m = 25,\ c = 10,\ k = 226$$

Char poly: $p(r) = mr^2 + cr + k$, roots $= \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m}$

Here: $c^2 - 4mk = 100 - 4 \times 25 \times 226 = -22500 = -(150)^2$

$$\Rightarrow \quad r = \frac{-10 \pm i\,150}{50}$$

$$\Rightarrow \quad x(t) = e^{-t/5}\left(A\cos 3t + B\sin 3t \right)$$

Using init. cond:

$$20 = x(0) = A\cos(0) + B\sin(0) = A$$

$$41 = x'(0) = -\frac{A}{5} + 3B \quad \Rightarrow B = 15$$

$$x'(t) = A\left(-3e^{-t/5}\sin 3t - \frac{1}{5}e^{-t/5}\cos 3t\right) + B\left(3e^{-t/5}\cos 3t - \frac{1}{5}e^{-t/5}\sin 3t\right)$$

$$\Rightarrow \boxed{x(t)} = e^{-t/5}\left(20\cos 3t + 15\sin 3t\right)$$

$$= e^{-t/5}\sqrt{20^2+15^2}\,\Big(\underbrace{\frac{20}{\sqrt{20^2+15^2}}}_{\cos\alpha}\cos 3t + \underbrace{\frac{15}{\sqrt{20^2+15^2}}}_{\sin\alpha}\sin 3t \Big)$$

$$= \boxed{25\,e^{-t/5}\cos(3t - \alpha)}$$

where $\alpha = \arccos\left(\frac{20}{25}\right) \approx 0.6435$

$\Rightarrow$ typical underdamped system as in Fig 3.4.15.

(b) • The pseudo angular freq $\omega = 3 \Rightarrow$ pseudoperiod $T = \frac{2\pi}{\omega} = \frac{2}{3}\pi$

The envelope curves are $\pm 25\,e^{-t/5}$.

3.4.16 The DE governing this system is:

$$mx'' + cx' + kx = 0$$

with $m = 3,\ c = 30,\ k = 63$

characteristic polynomial:

$$p(r) = mr^2 + cr + k \Rightarrow \text{roots} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

here: $c^2 - 4mk = 900 - 4 \times 3 \times 63 = 900 - 756 = 144 > 0$

$\Rightarrow$ system is <u>overdamped</u>.

roots are $r = \dfrac{-30 \pm 12}{6} = \begin{cases} -7 \\ -3 \end{cases}$

thus $x(t) = Ae^{-7t} + Be^{-3t}$

Using initial conditions:

$$\left.\begin{aligned} 2 &= x(0) = A + B \\ 2 &= v(0) = x'(0) = -7A - 3B \end{aligned}\right\} \Rightarrow \begin{aligned} A &= -2 \\ B &= 4 \end{aligned}$$

$$\Rightarrow \boxed{x(t) = -2e^{-7t} + 4e^{-3t}}$$

<u>In the undamped case</u>: $mx'' + kx = 0$

$\Rightarrow x(t) = A\cos\omega_0 t + B\sin\omega_0 t$

with $\omega_0 = \sqrt{\dfrac{k}{m}} = \sqrt{21}$

Using init conditions:

$$\left.\begin{aligned} 2 &= x(0) = A \\ 2 &= x'(0) = \omega_0 B \end{aligned}\right] \Rightarrow x(t) = 2\cos\sqrt{21}\,t + \frac{2}{\omega_0}\sin\sqrt{21}\,t$$

$$= 2\sqrt{\frac{22}{21}}\cos\left(\sqrt{21}\,t - \alpha\right).$$

amplitude is: $C = \sqrt{2^2 + \left(\dfrac{2}{\omega_0}\right)^2} = \sqrt{4 + \dfrac{4}{21}} = 2\sqrt{1 + \dfrac{1}{21}} = 2\sqrt{\dfrac{22}{21}}$; $\alpha = \cos^{-1}\left(\dfrac{2}{C}\right) = 0.21485$

4.16 cont'd

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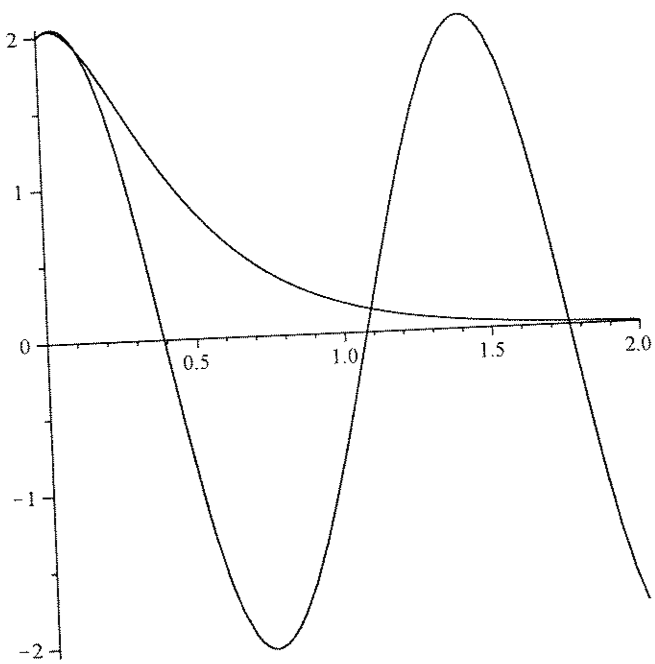
> x1 := t-> -2*exp(-7*t) + 4*exp(-3*t);
x2 := t-> 2*sqrt(22/21) * cos(sqrt(21)*t - arccos(sqrt(21/22)));
plot({x1(t),x2(t)},t=0..2,color=black);
```

$$x1 := t \to -2\,e^{-7t} + 4\,e^{-3t}$$

$$x2 := t \to 2\sqrt{\frac{22}{21}}\cos\left(\sqrt{21}\,t - \arccos\left(\sqrt{\frac{21}{22}}\right)\right)$$



3.5.3 (optional)

Find a particular solution to:

$$L(y) = y'' - y' - 6y = 2\sin 3x = f(x)$$

here characteristic polyn. is $p(r) = r^2 - r - 6 = (r-3)(r+2)$

$\Rightarrow$ $\sin 3x \notin \operatorname{Ker} L$ so our guess of y_p (search space) can be:

$$V = \operatorname{span}\{\cos 3x, \sin 3x\}, \text{ Basis } \mathcal{B} = \{\cos 3x, \sin 3x\}.$$

in this basis of V: $(f)_{\mathcal{B}} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

.. 5.3
ont'd

And L in the same basis is:

$$(L)_{\mathcal{B}} = \left[(L(\cos 3x))_{\mathcal{B}} \quad (L(\sin 3x))_{\mathcal{B}} \right]$$

$$= \begin{bmatrix} -15 & -3 \\ 3 & -15 \end{bmatrix}$$

Since:

$$L(\cos 3x) = -9\cos 3x + 3\sin 3x - 6\cos 3x$$
$$= -15\cos 3x + 3\sin 3x$$
$$\equiv \begin{pmatrix} -15 \\ 3 \end{pmatrix} \text{ in the basis } \mathcal{B}$$

$$L(\sin 3x) = -9\sin 3x - 3\cos 3x - 6\sin 3x$$
$$= -3\cos 3x - 15\sin 3x$$
$$\equiv \begin{pmatrix} -3 \\ -15 \end{pmatrix} \text{ in the basis } \mathcal{B}.$$

Thus to find $y_p = A\cos 3x + B\sin 3x$ we need to solve system:

$$\begin{bmatrix} -15 & -3 \\ 3 & -15 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{234} \begin{bmatrix} -15 & 3 \\ -3 & -15 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6/234 \\ -30/234 \end{bmatrix} = \begin{bmatrix} 1/39 \\ -5/39 \end{bmatrix}$$

$$\Rightarrow \boxed{y_p = \frac{1}{39}(\cos 3x - 5\sin 3x)}$$

3.5.5 $L(y) = y'' + y' + y = \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x = f(x)$

recall: $\cos(2a) = \cos^2 a - \sin^2 a = 1 - 2\sin^2 a$

$\Rightarrow \sin^2 a = \dfrac{1-\cos 2a}{2}$

We have: $L(1) = 1$

$L(\cos 2x) = -4\cos 2x - 2\sin 2x + \cos 2x \neq 0$

$L(\sin 2x) = -4\sin 2x + 2\cos 2x + \sin 2x \neq 0$

So we can use as search space $V = \text{span}\{\underbrace{1, \cos 2x, \sin 2x}_{\mathcal{B}}\}$.

In the basis $\mathcal{B}$ we have:

$$(f)_{\mathcal{B}} = \underbrace{\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix}}_{b} \qquad (L)_{\mathcal{B}} = \left[(L(1))_{\mathcal{B}} \; (L(\cos 2x))_{\mathcal{B}} \; (L(\sin 2x))_{\mathcal{B}} \right]$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & -2 & -3 \end{bmatrix}}_{A}$$

Thus to find $y_p = c_1 + c_2 \cos 2x + c_3 \sin 2x$ we solve the system

$$A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = b. \quad \text{By inspection } \boxed{c_1 = \frac{1}{2}}$$

and $$\begin{pmatrix} c_2 \\ c_3 \end{pmatrix} = \begin{bmatrix} -3 & 2 \\ -2 & -3 \end{bmatrix}^{-1} \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} -3 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}$$

$$= \frac{1}{26} \begin{bmatrix} 3 \\ -2 \end{bmatrix} \Rightarrow \boxed{y_p = \frac{1}{2} + \frac{1}{26}(3\cos 2x - 2\sin 2x).}$$

3.5.7 $L(y) = y'' - 4y = \sinh x = \dfrac{e^{x} - e^{-x}}{2} = f(x)$

Our search space will be:

$$V = \operatorname{span}\underbrace{\{e^{x}, e^{-x}\}}_{\mathcal{B}}.$$

We have: $(f)_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = b$

and $(L)_{\mathcal{B}} = \left[(Le^{x})_{\mathcal{B}} \quad (Le^{-x})_{\mathcal{B}} \right]$

$$= \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} = A$$

since
$$L(e^{x}) = e^{x} - 4e^{x} = -3e^{x}$$
$$L(e^{-x}) = e^{-x} - 4e^{x} = -3e^{-x}$$

thus to find $y_p = c_1 e^{x} + c_2 e^{-x}$ we solve:

$$A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = b \implies \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = A^{-1}b = \begin{bmatrix} -1/3 & 0 \\ 0 & -1/3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1/6 \\ 1/6 \end{bmatrix}$$

$$\implies \boxed{y_p = \frac{1}{6}\left(-e^{x} + e^{-x}\right) = -\frac{1}{3}\sinh x}.$$

5.32 Solve initial value problem:

$$\begin{cases} y'' + 3y' + 2y = e^x = f(x) \\ y(0) = 1 \\ y'(0) = 3 \end{cases}$$

Characteristic poly of $L(y) = y'' + 3y' + 2y$ is:

$$p(r) = r^2 + 3r + 2 = (r+1)(r+2)$$

$\Rightarrow y_H = c_1 e^{-x} + c_2 e^{-2x}$

Our guess for $y_p = A e^x$

$$L(y_p) = Ae^x + 3Ae^x + 2Ae^x = e^x$$

$$\Rightarrow \quad 6A = 1 \quad \Rightarrow \quad A = \frac{1}{6}$$

$$\Rightarrow \quad y_p = \frac{1}{6} e^x$$

$$\Rightarrow \quad y = \underbrace{\frac{1}{6} e^x}_{y_p} + \underbrace{c_1 e^{-x} + c_2 e^{-2x}}_{y_H}$$

Using I.C.:

$$0 = y(0) = \frac{1}{6} + c_1 + c_2$$

$$3 = y'(0) = \frac{1}{6} - c_1 - 2c_2$$

$$\Rightarrow c_1 = \frac{5}{2} \qquad c_2 = -\frac{8}{3}$$

$$\boxed{\Rightarrow y = \frac{1}{6} e^x + \frac{5}{2} e^{-x} - \frac{8}{3} e^{-2x}}$$

3.5-35

$$\begin{cases} y'' - 2y' + 2y = x + 1 = f(x) \\ y(0) = 3 \\ y'(0) = 0 \end{cases}$$

Char poly of $L(y) = y'' - 2y' + 2y$ is $p(r) = r^2 - 2r + 2$

it has roots $r = \dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

$\Rightarrow y_H = A e^x \cos x + B e^x \sin x$

To find y_P we use search space $= V = \{\underbrace{1, x}_{\mathcal{B}}\}$

$(f)_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $(L)_{\mathcal{B}} = [(L(1))_{\mathcal{B}} (L(x))_{\mathcal{B}}]$

$$= \begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}$$

$L(1) = 0 - 0 + 2 = 2$

$L(x) = 0 - 2 + 2x = -2 + 2x$

$\Rightarrow y_P = c_1 + c_2 x$ can be obtained by solving

$$\begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

$\left.\begin{array}{l} (e^x \cos x)' = e^x(\cos x - \sin x) \\ (e^x \sin x)' = e^x(\sin x + \cos x) \end{array}\right\}$ $\Rightarrow y_P = 1 + \dfrac{x}{2}$ $\Rightarrow y = y_P + y_H = 1 + \dfrac{x}{2} + Ae^x \cos x + Be^x \sin x$

Using initial conditions:

$3 = y(0) = 1 + A \Rightarrow A = 2$

$0 = y'(0) = \dfrac{1}{2} + A + B \Rightarrow B = -5/2$

$$\Rightarrow \boxed{y(x) = 1 + \frac{x}{2} + e^x\left(2\cos x - \frac{5}{2}\sin x\right)}$$