HW3 MATH2280 Solutions.

2.3.2 (a) Body moves according to $\frac{dv}{dt} = -kv \implies v(t) = v_0 e^{-kt}$ using e.g. separation of variables.

By integration: $\boxed{x(t) = x_0 + \int_0^t v(s)\,ds = x_0 + \frac{v_0}{-k} e^{-ks}\Big|_0^t = x_0 + \frac{v_0}{k}(1 - e^{-kt})}$

(b) We have $\lim_{t\to\infty} x(t) = x_0 + \frac{v_0}{k} < \infty$

$=$ position after a long time.

distance travelled is then v_0/k

2.3.3 optional)

We know $v(0) = 40$ (ft/s)

$v(10) = 20$ (ft/s)

$= 40 e^{-k10}$ (velocity is $v(t) = v_0 e^{-kt}$, see prev. problem)

$\Rightarrow -k10 = \ln\frac{1}{2}$

$\Rightarrow k = \frac{\ln 2}{10}$

Thus distance traveled by boat is $\frac{v_0}{k} = \frac{400}{\ln 2} \approx \boxed{577\text{ft}}$

2.3.4 Body moves according to $\frac{dv}{dt} = -kv^2$

sep of vars: $\int -\frac{dv}{v^2} = \int dt + C \Rightarrow v(t) = \frac{1}{t+c}$

and $v(0) = \frac{1}{c} \Rightarrow c = 1/v_0 \Rightarrow \boxed{v(t) = \frac{1}{t+1/v_0} = \frac{v_0}{tv_0+1}}$

3.4 cont'd

Now by integration:

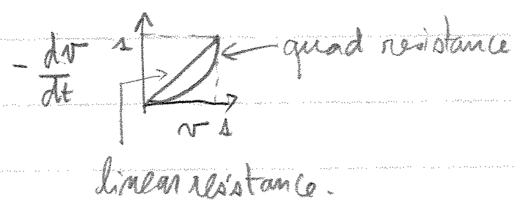
$$\boxed{x(t) = x_0 + \int_0^t v(s)\,ds} = x_0 + \int_0^t \frac{v_0}{1+v_0 k s}\,ds = x_0 + \frac{v_0}{v_0 k}\ln(1+v_0 k s)\Big|_0^t$$

$$\boxed{= x_0 + \frac{1}{k}\ln(1+v_0 k t)}$$

$x(t) \to \infty$ as $t \to \infty$.

When object moves slowly: $v(t)$ is small say $v(t) < 1$

in this case $v^2 \leq v$ $\Rightarrow$ less resistance in quadratic case for small velocities, so since as $t \to \infty$ both velocities $\to 0$, less resistance means more distance traveled.



2.3.11 Using eq (4) we get $v_z = -\frac{g}{\rho} \Rightarrow \rho = -g/v_z$

$$= \frac{-32\,ft.s^{-2}}{-100\,mph}$$

We will put all quantities in feets, seconds so: $1\,mph \approx \frac{88}{60}\,f/s$

$$\Rightarrow \rho \approx \frac{32 \times 60}{8800} \approx 0.218$$

Now using eq. (9) we can estimate time T it takes to drop 1200 ft:

$$0 = y(T) = y(0) + v_z T + \frac{1}{\rho}(v_0 - v_z)(1 - e^{-\rho T})$$

where $y(0) = 1200\,ft$, $v_z \approx -146.7\,ft/s$, $v_0 = 0\,ft/s$ and

2.3.11 cont'd

We solve this (transcendental) equation using maple to get:

$$T \approx 12.5s$$

So the report is not very accurate, the paratrooper would have impacted land at a velocity < 100mph, which explains maybe why he/she survived.

2.3.13

Separation of variables on DE (12):

$$\frac{dv}{dt} = -g\left(1 + \frac{\rho}{g} v^2\right) \text{ gives:}$$

$$\int \frac{dv}{1 + \frac{\rho}{g} v^2} = -\int g\,dt + C_1$$

$$\parallel \qquad v = u\sqrt{\frac{g}{\rho}} \Rightarrow dv = du\sqrt{\frac{g}{\rho}}$$

$$\int \frac{du\sqrt{g/\rho}}{1 + u^2} = \sqrt{g/\rho}\,\text{atan}(u)$$

$$\Rightarrow \quad \text{atan}(u) = -\sqrt{\frac{\rho}{g}}\,gt + C_1$$

$$\Rightarrow \quad u = \tan\left(-\sqrt{g\rho}\,t + C_1\right)$$

$$\Rightarrow \quad v = \sqrt{\frac{g}{\rho}}\tan\left(-\sqrt{g\rho}\,t + C_1\right)$$

$$v_0 = v(0) = \sqrt{\frac{g}{\rho}}\tan(C_1)$$

$$\Rightarrow C_1 = \text{atan}\left(v_0\sqrt{\frac{\rho}{g}}\right) \quad \text{QED.}$$

.3.14

(optional) $y(t) = y_0 + \int_0^t v(s)\,ds = y_0 + \int_0^t \sqrt{\frac{g}{\rho}} \tan(C_1 - s\sqrt{\rho g})\,ds$

Let $u = C_1 - s\sqrt{\rho g}$

$du = -ds\sqrt{\rho g}$

$$\Rightarrow \boxed{y(t)} = y_0 + \int_{C_1}^{C_1 - t\sqrt{\rho g}} \underbrace{-\sqrt{\frac{g}{\rho}}\frac{1}{\sqrt{\rho g}}}_{-1/\rho} \tan(u)\,du$$

$$= y_0 + \frac{1}{\rho}\ln|\cos u| \Big|_{C_1}^{C_1 - t\sqrt{\rho g}}$$

$$\boxed{= y_0 + \frac{1}{\rho}\ln\left|\frac{\cos(C_1 - t\sqrt{\rho g})}{\cos C_1}\right|}$$ QED

2.3.15 Calculation is line by line the same as 2.3.13:
Using sep of vars on DE: (15):

$$\frac{dv}{dt} = -g\left(1 - \frac{\rho}{g}v^2\right)$$

$$\int \frac{dv}{1 - \frac{\rho}{g}v^2} = -\int g\,dt + C_2 = -gt + C_2$$

$\parallel$ $\quad v = u\sqrt{g/\rho} \Rightarrow dv = du\sqrt{g/\rho}$

$$\int \frac{du\sqrt{g/\rho}}{1 - u^2} = \sqrt{g/\rho}\ \mathrm{atanh}(u)$$

$$\Rightarrow u = \tanh(C_2 - \sqrt{g\rho}\,t)$$

2.3.15 cont'd

$$\Rightarrow v = \sqrt{\frac{g}{\rho}} \tanh(C_2 - \sqrt{g\rho}\, t)$$

at $v(0) = v_0 = \sqrt{\frac{g}{\rho}} \tanh C_2 \Rightarrow C_2 = \operatorname{atanh}\left(v_0 \sqrt{\frac{\rho}{g}}\right)$.

QED.

2.3.16 (optional)

The calculations here are analogous to the ones in 2.3.14:

$$y(t) = y_0 + \int_0^t v(s)\, ds = y_0 + \int_0^t \sqrt{\frac{g}{\rho}} \tanh(C_2 - s\sqrt{\rho g})\, ds$$

let $u = C_2 - s\sqrt{\rho g} \Rightarrow du = -ds\sqrt{\rho g}$

$$\Rightarrow y(t) = y_0 + -\frac{1}{\rho} \int_{C_2}^{C_2 - t\sqrt{\rho g}} \tanh(u)\, du$$

$$= y_0 - \frac{1}{\rho} \ln|\cosh u| \Big|_{C_2}^{C_2 - t\sqrt{\rho g}}$$

$$= y_0 - \frac{1}{\rho} \ln\left|\frac{\cosh(C_2 - t\sqrt{\rho g})}{\cosh C_2}\right|.$$

QED.

2.3.17

We have $v_0 = 49\, m/s$, $g = 9.8\, m/s^2$, $\rho = 0.0011$.

To find max height we first need to use eq. (13) to find time T at which $v(T) = 0$.

By inspection:

$$T = \frac{C_1}{\sqrt{\rho g}}; \text{ where } C_1 = \operatorname{atan}\left(v_0 \sqrt{\frac{\rho}{g}}\right) \approx 0.48$$

$$\approx 4.61 s$$

2.3.17 Cont'd

Plugin value of T in eq (14) we get:

$$\boxed{y(T) = \cancel{y_0} + \frac{1}{\rho}\ln\left|\frac{\cos C_1 - \cancel{T\sqrt{\rho g}}}{\cos C_1}\right|}$$

$$= \frac{1}{\rho}\ln\left|\frac{1}{\cos C_1}\right| \simeq \boxed{108.47\,m}$$

(max height)

2.3.18 optional)

We have now $v_0 = 0$, $y_0 = 108.47m$

$$C_2 = \operatorname{arctanh}\left(v_0\sqrt{\frac{\rho}{g}}\right) = \operatorname{arctanh}(0) = 0$$

then we use (17) to find time T for which $y(T)=0$:

$$0 = y(T) = y_0 - \frac{1}{\rho}\ln\left|\frac{\cosh(-T\sqrt{\rho g})}{\cancel{\cosh(0)}\ 1}\right|$$

$$\Rightarrow \quad \cosh(T\sqrt{\rho g}) = \exp(\rho y_0)$$

$$\Rightarrow \quad \boxed{T = \frac{1}{\sqrt{\rho g}}\operatorname{acosh}(\exp(\rho y_0))}$$

$$\boxed{\approx 4.8s}$$

To find landing velocity we look at (16):

$$\boxed{v(T) = \sqrt{\frac{g}{\rho}}\tanh(\cancel{C_2} - T\sqrt{\rho g}) = -43.49\,m/s}$$