1.5.5 We need to solve:

$$\begin{cases} xy' + 2y = 3x \\ y(1) = 5 \end{cases}$$

We find first a general solution to

$$y' + \frac{2}{x} y = 3 \qquad (x \neq 0)$$

using the integrating factor

$$e^{2 \ln|x|} = x^2.$$

$$\underbrace{x^2\left(y' + \frac{2}{x} y\right)}_{= (x^2 y)'} = 3x^2 \implies \boxed{y = x + \frac{C}{x^2}} = \text{general solution}$$

The particular solution for which $y(1) = 5$ is:

$$y(1) = 5 = 1 + \frac{C}{1^2} \implies C = 4$$

$$\boxed{y(x) = x + \frac{4}{x^2}}$$

1.5.13 We need to solve

$$\begin{cases} y' + y = e^x \\ y(0) = 1 \end{cases}$$

We first find a general solution using integrating factor method:

$$\underbrace{e^x (y' + y)}_{= (e^x y)'} = e^{2x}$$

Thus a general solution is:

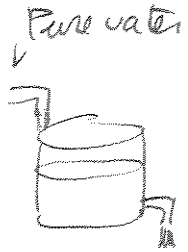
$$\boxed{y(x) = \frac{e^x}{2} + C e^{-x}}$$

And the particular solution s.t. $y(0) = 1$:

$$y(0) = 1 = \frac{1}{2} + C \implies C = \frac{1}{2}$$

$$\implies \boxed{y(x) = \frac{e^x + e^{-x}}{2} = \cosh(x)}$$

1.5.33)



Since $r_i = r_o = r = 5\ell/s$ volume in tank remains constant.

$$V(t) = V(0) = V = 1000\,\ell.$$

Let $x(t)$ be the quantity of salt at time t contained in tank (kg) then:

$$\begin{cases} \dfrac{dx}{dt} = r c_i - r\dfrac{x}{V} \\ x(0) = 100 \end{cases}$$

where $c_i = 0$ because only pure water enters tank

Solution: $x(t) = x(0)\, e^{-\frac{r}{V}t}$

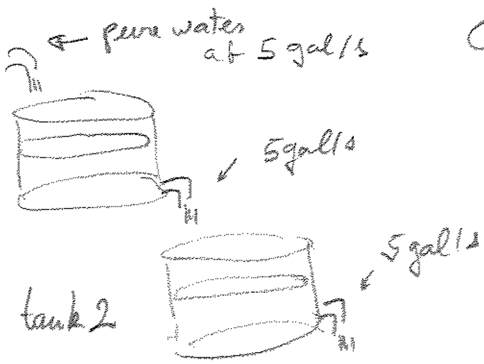
Thus we now find T s.t. $x(T) = 10\text{kg} = 100\text{kg}\, e^{-\frac{r}{V}T}$

$$\Rightarrow \frac{1}{10} = e^{-\frac{5}{1000}T}$$

$$\boxed{T = \left(\frac{-1000}{5}\right)(-\ln 10)}$$

$$\simeq 460.52\,s \simeq \boxed{7\text{min } 41\,s}$$

1.5.38)



Cascade of two tanks

$V_1 = 100$ gal

$V_2 = 200$ gal

$x(t) =$ amount of salt (lb) in tank 1; $x(0) = 50$.

$y(t) =$ amount of salt (lb) in tank 2; $y(0) = 50$.

Concentration in $= 0$ (pure water)

$r = 5\text{gal/s} =$ rate in tank 1
$=$ rate out tank 1
$=$ rate out tank 2

The tank volumes remain <u>constant</u>.

(a) DE satisfied by quantity of salt in Tank 1 is:

$$\begin{cases} \dfrac{dx}{dt} = \cancel{rc}^{\,0} - r\dfrac{x}{V_1} \\ x(0) = 50 \end{cases} \Rightarrow \boxed{x(t) = x(0)\, e^{-\frac{r}{V_1}t}}$$

$$= 50\, e^{-t/20}$$

(b) rate out tank 1 = rate in tank 2 = 5 gal/s
concentration out tank 1 = concentration in tank 2
concentration out tank 2 = $y(t)/V_2$

Thus we get DE:

$$\frac{dy}{dx} = rc_i - rc_o = r\frac{x}{V_1} - r\frac{y}{V_2} = \frac{5x}{100} - \frac{5y}{200} = \frac{x}{20} - \frac{y}{40}$$

So y solves

$$\begin{cases} \frac{dy}{dx} = \frac{x}{20} - \frac{y}{40} \\ y(0) = 50 \end{cases}$$

which can be solved using integ. factor:

$$e^{t/40}\left(y' + \frac{1}{40}y\right) = e^{t/40}\frac{1}{20}50e^{-t/20}$$

$$\left(e^{t/40}y\right)' = \frac{5}{2}e^{-t/40}$$

$$\Rightarrow \quad e^{t/40}y = -\frac{5}{2}\cdot 40\, e^{-t/40} + C$$

$$\Rightarrow \quad y(t) = -100e^{-t/20} + Ce^{-t/40}$$

And using $y(0) = 50 = -100 + C \Rightarrow C = 150$

$$\Rightarrow \quad \boxed{y(t) = -100e^{-t/20} + 150e^{-t/40}}$$

(c) To find largest amount of salt in tank 2 we look at extrema of $y(t)$ ie points for which:

$$0 = y'(t) = (-100)(-1/20)e^{-t/20} + (150)(-1/40)e^{-t/40}$$

$$\Leftrightarrow \quad 0 = 5e^{-t/20} - \frac{15}{4}e^{-t/40}$$

$$\Leftrightarrow \quad e^{-t/40} = \frac{15}{4\cdot 5} = \frac{3}{4}$$

$$\Leftrightarrow \quad t_{max} = -40\ln\frac{3}{4} \simeq 11.507\ s$$

Thus max ~~amount~~ is $y(t_{max}) \simeq 56.25$ lb

5.41

$S(t) = 30\, e^{t/20}$ = salary in 1k\$.

$A(t)$ = amount in retirement account

$r = 6\%$ = rate of interest of retirement account

(a) In a time Δt :

- $12\%\ S(t)\,\Delta t$ is added to account from salary
- $r\,A(t)\,\Delta t$ is added to account from interest

$\Rightarrow \quad \Delta A = 12\%\ S(t)\,\Delta t + r\,A(t)\,\Delta t$

Thus $A(t)$ satisfies DE (as $\Delta t \to 0$).

$$\begin{cases} \dfrac{dA}{dt} = rA + \dfrac{12}{100} S(t). = \dfrac{6}{100} A + \dfrac{12}{100} \times 30 e^{t/20} \\ \qquad\qquad\qquad\qquad = \dfrac{6}{100} A + \dfrac{360}{100} e^{t/20} \\ A(0) = 0 \qquad \text{(retirement account is empty)} \end{cases}$$

We solve this equation using integ. factor $e^{-6/100\, t}$:

$$\left(e^{-6t/100} A \right)' = \frac{360}{100} e^{t/20}\, e^{-6t/100} = \frac{360}{100} e^{-t/100}$$

$$\Rightarrow e^{-6t/100} A = \frac{360(-100)}{100} e^{-t/100} + C$$

$$\Rightarrow \qquad A(t) = -360\, e^{5t/100} + C\, e^{6t/100}$$

and $\quad 0 = A(0) = -360 + C \Rightarrow C = 360$

$$\Rightarrow \quad \boxed{A(t) = 360 \left[-e^{5t/100} + e^{6t/100} \right]}$$

b) After 40 y the amount in account is:

$A(40) \approx 1\,308.283$ thousand dollars

So the account will have approx $\boxed{\$1\,308\,283}$.

2.1.3) Solve:

$$\begin{cases} \dfrac{dx}{dt} = 1 - x^2 \\ x(0) = 3 \end{cases}$$

Sep of vars:

$$\int \frac{dx}{1-x^2} = \int dt + C$$

We have:

$$\int \frac{dx}{1-x^2} = \int \frac{dx}{(1-x)(1+x)} = \int \left(\frac{1/2}{1-x} + \frac{1/2}{1+x} \right) dx$$

$$= -\frac{1}{2} \ln |1-x| + \frac{1}{2} \ln |1+x|$$

For the particular initial value: $x - 1 > 0$ and $x + 1 > 0$.

$$\Rightarrow \quad \frac{1}{2} \ln \frac{1+x}{-1+x} = t + C$$

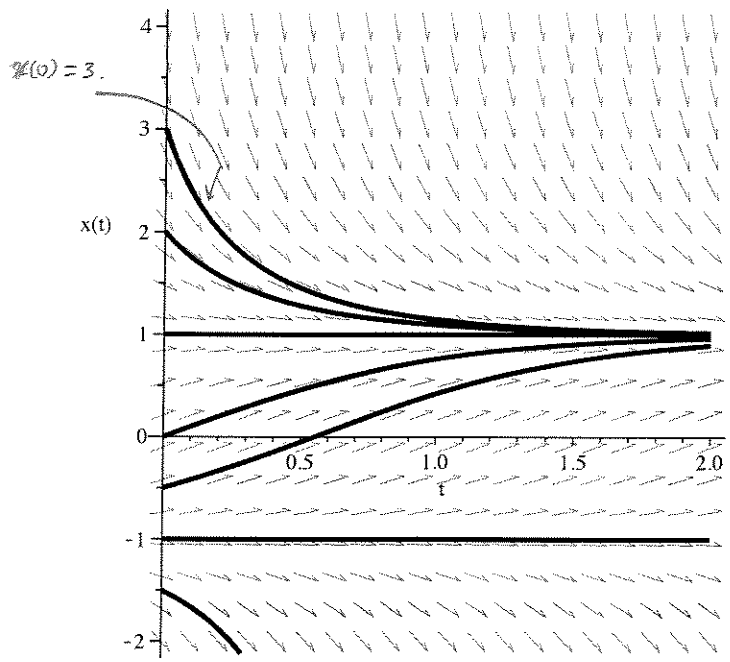
$$\Rightarrow \quad \frac{1+x}{-1+x} = e^{2t+C} = Ae^{2t}$$

Since $x(0) = 3$: $A = \frac{3+1}{3-1} = 2$

and $\boxed{x(t) = \dfrac{2e^{2t}+1}{2e^{2t}-1}} = \boxed{\dfrac{2+e^{-2t}}{2-e^{-2t}}}$

As expected $x(t) \to 1$ as $t \to \infty$. The other equilibrium solution should be $x(t) = -1$ (homogeneous DE).

```
> with(DEtools):
  DEplot(diff(x(t),t)=1-x(t)^2,x(t),t=0..2,{x(0)=3,x(0)=2,x(0)=1,x
  (0)=0,x(0)=-0.5,x(0)=-1,x(0)=-1.5},x=-2..4,linecolor=black);
```



>

-1.4 Solve $\begin{cases} \dfrac{dx}{dt} = 9 - 4x^2 \\ x(0) = 0 \end{cases}$ Sep of vars:

$$\int \frac{dx}{9-4x^2} = \int dt + C$$

We have $\displaystyle\int \frac{dx}{9-4x^2} = \int \frac{dx}{(3-2x)(3+2x)} = \int \left(\frac{1/6}{3-2x} + \frac{1/6}{3+2x} \right) dx$

$$= -\frac{1}{12} \ln|3-2x| + \frac{1}{12} \ln|3+2x|$$

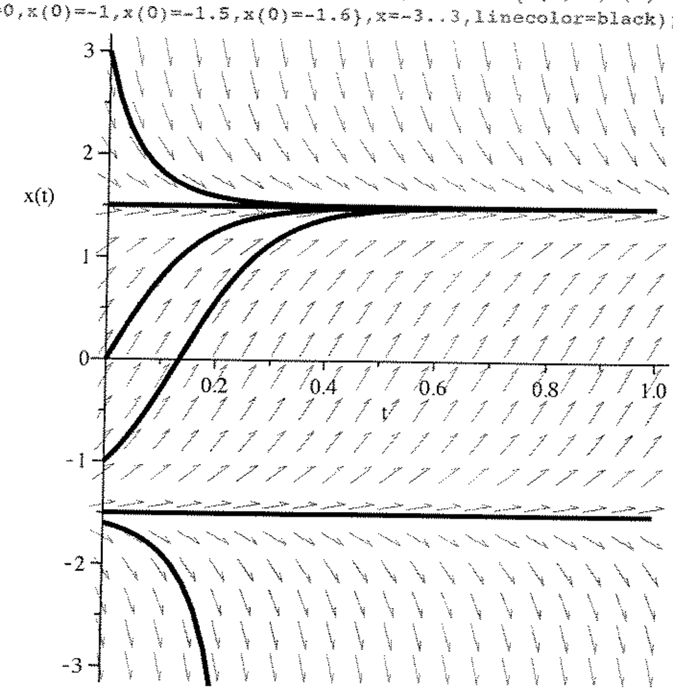
For this particular solution: $3-2x > 0$ and $3+2x > 0$.

$$\Rightarrow \quad \frac{1}{12} \ln \frac{3+2x}{3-2x} = t + C$$

$$\Rightarrow \quad \frac{3+2x}{3-2x} = Ae^{12t} \quad \Rightarrow \quad A = 1 \text{ since } x(0) = 0.$$

$$\Rightarrow \quad \boxed{x(t) = \frac{3}{2} \frac{e^{12t} - 1}{e^{12t} + 1} = \frac{3}{2} \frac{1 - e^{-12t}}{1 + e^{-12t}}}$$

```
> with(DEtools):
  DEplot(diff(x(t),t)=9-4*x(t)^2,x(t),t=0..1,{x(0)=3,x(0)=1.5,x(0)
  =0,x(0)=-1,x(0)=-1.5,x(0)=-1.6},x=-3..3,linecolor=black);
```



>

2.1.13 (a) The DE satisfied by the rabbit population is:

$$\frac{dP}{dt} = (\beta - \delta) P.$$ We know $\beta = bP$, $b > 0$ constant and $\delta = dP$, $d > 0$ constant

$$= (b-d) P^2$$
$$= k P^2$$ where $k = b - d > 0$ since $\beta > \delta$.

$$\Rightarrow \int \frac{dP}{P^2} = \int k\,dt + C$$

$$\Rightarrow (*) \quad -\frac{1}{P} = kt + C \Rightarrow C = -\frac{1}{P_0}$$ substituting $t = 0$.

and thus:

$$\boxed{P = \frac{-1}{kt - \frac{1}{P_0}} = \frac{P_0}{1 - P_0 k t}}$$

(b) If $P_0 = 6$ and $P(10) = 9$ we can find k:

$$-\frac{1}{P(10)} = -\frac{1}{9} = k\,10 - \frac{1}{6}$$ using $(*)$

$$\Rightarrow k = \frac{1}{10}\left(\frac{1}{6} - \frac{1}{9}\right) = \frac{1}{180}$$

$$\Rightarrow \text{doomsday} = t = \frac{1}{1/180\ 6} = 30 \text{ months}$$

2.1.15 The population satisfies DE:

$$\frac{dP}{dt} = aP - bP^2 = bP\left(\frac{a}{b} - P\right)$$

$$\rightarrow \text{limiting population is } \frac{a}{b} = \frac{aP_0^2}{bP_0^2} = \frac{B_0 P_0}{D_0}$$

2.1.22 The key is that the population having heard rumor follows logistic equation:

$$\begin{cases} \dfrac{dP}{dt} = k P (M - P) \\ P(0) = \dfrac{10^5}{2} = 50000 \end{cases}$$

where the limiting pop $M = 100\,000 = 2P_0$

To find k we use the additional piece of info that:

$$10^3 = P'(0) = k\,P(0)\,(M - P(0)) = k\,P_0^2$$

$$\Rightarrow \quad k = \frac{10^3}{(10^5/2)^2} = 4 \cdot 10^{-7}$$

We now use the solution to logistic equation:

$$P(t) = \frac{M P_0}{P_0 + (M - P_0)e^{-kMt}} \qquad \text{(eq (6) in textbook)}$$

to find time T at which 80% of population has heard rumor:

$$2P_0 \times 0.8 = M \times 0.8 = P(T) = \frac{M P_0}{P_0 + (M-P_0)\,e^{-kMT}} = \frac{2P_0^{\cancel{2}}}{\cancel{P_0} + \cancel{P_0}\,e^{-2kP_0T}}$$

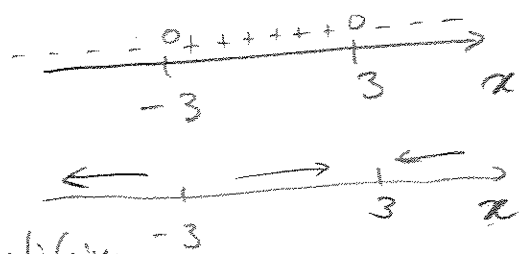
$$\Rightarrow \quad 1 + e^{-2kP_0T} = \frac{1}{0.8} \quad \Rightarrow \quad \boxed{T = \frac{-1}{2kP_0}\log\left(\frac{0.2}{0.8}\right) \simeq 34.66 \text{ days}}$$

2.2.6

$$\frac{dx}{dt} = 9 - x^2$$

Solving $9 - x^2 = 0$ we get two equilibria $x = 3$ and $x = -3$.

The phase diagram is:



so $x = -3$ should be an unstable equilibrium

$x = 3$ ——— a stable ———

We solve the DE explicitly using separation of variables to obtain:

$$\int \frac{dx}{9 - x^2} = \int dt + C$$

$$\int \left(\frac{1/6}{3-x} + \frac{1/6}{3+x}\right) dx = \frac{1}{6}\ln\frac{|3+x|}{|3-x|}$$

$$\Rightarrow \begin{cases} \dfrac{3+x}{3-x} = Ae^{6t} & \text{if } 9 - x^2 \geq 0 \\[2ex] \dfrac{3+x}{x-3} = Ae^{6t} & \text{otherwise} \end{cases}$$

2.2.6 cont'd This is equivalent to:

$$(*) \quad \frac{3+x}{3-x} = A e^{6t} \quad \text{with } A \neq 0$$

$$\Rightarrow \quad \boxed{x(t) = \frac{3(Ae^{6t}-1)}{Ae^{6t}+1} = \frac{3(A-e^{-6t})}{(A+e^{-6t})}}$$

By letting $t=0$ in $(*)$ we get $A = \frac{3+x_0}{3-x_0}$

thus another expression for $x(t)$ is:

$$\boxed{x(t) = \frac{3(3+x_0-(3-x_0)e^{-6t})}{3+x_0+(3-x_0)e^{-6t}}}$$

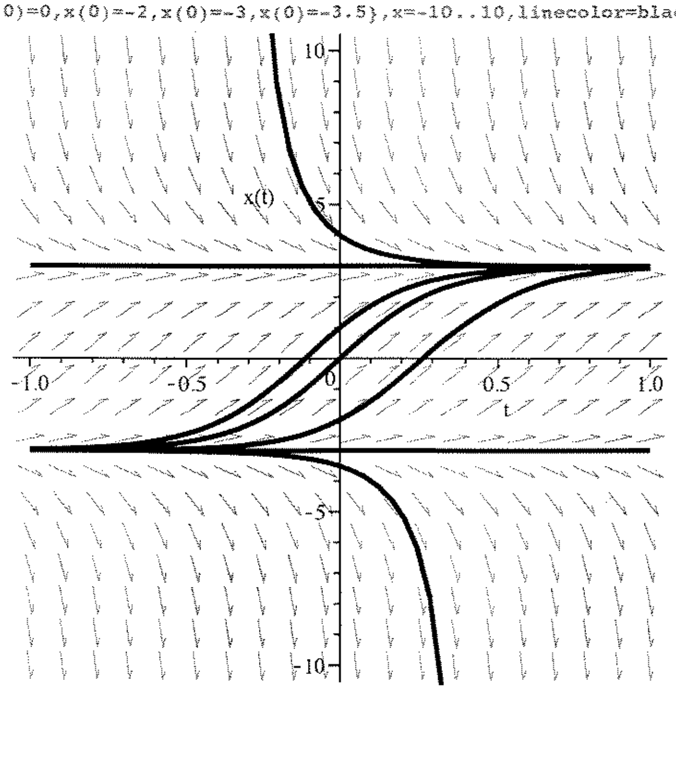
We can assert that $x(t) \to 3$ as $t \to \infty$ as long as the denominator does not become 0. You can check that this does happen only when $x_0^2 - 9 \geq 0$, that is when $x_0 \in]-\infty,-3[\cup]3,\infty[$

For $x_0 \geq 3$ $\quad x(t) \to \infty$ as $t \to t_0$, some negative doomsday time

$x_0 \leq -3$ $\quad x(t) \to -\infty$ as $t \to t_0$, some positive " "

$-3 \leq x_0 \leq 3 \Rightarrow x(t) \to 3$ as $t \to \infty$

```
> with(DEtools):
  DEplot(diff(x(t),t)=9-x(t)^2,x(t),t=-1..1,{x(0)=4,x(0)=3,x(0)=1,
  x(0)=0,x(0)=-2,x(0)=-3,x(0)=-3.5},x=-10..10,linecolor=black);
```




```
>
```

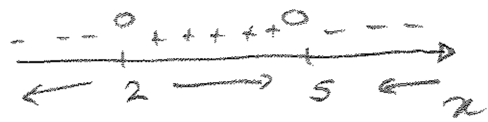
(10)

2.2.10: $\dfrac{dx}{dt} = 7x - x^2 - 10$

Solving $-x^2 + 7x - 10 = 0$ we get:

$x = 2$ or $x = 5$.

The phase diagram is:



5: stable equillibri

2: unstable.

By separation of variables:

$$\int \frac{dx}{7x - x^2 - 10} = \int dt + C = t + C$$

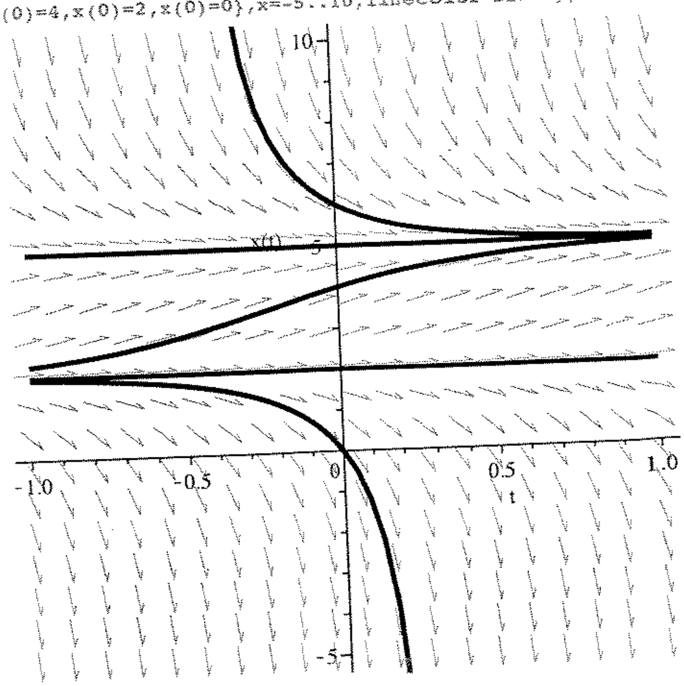
$$\int \frac{dx}{-(x-2)(x-5)} = \int \left(\frac{1/3}{x-2} - \frac{1/3}{x-5} \right) dx$$

$$\Rightarrow \quad \frac{1}{3} \ln \frac{|x-2|}{|x-5|} = t + C$$

$$\Rightarrow \quad \frac{x-2}{x-5} = Ae^{3t} \quad \text{for } A \neq 0$$

$$\Rightarrow \quad \boxed{x(t) = \frac{5Ae^{3t} - 2}{Ae^{3t} - 1} \quad \text{with } A = \frac{x_0 - 2}{x_0 - 5}.}$$

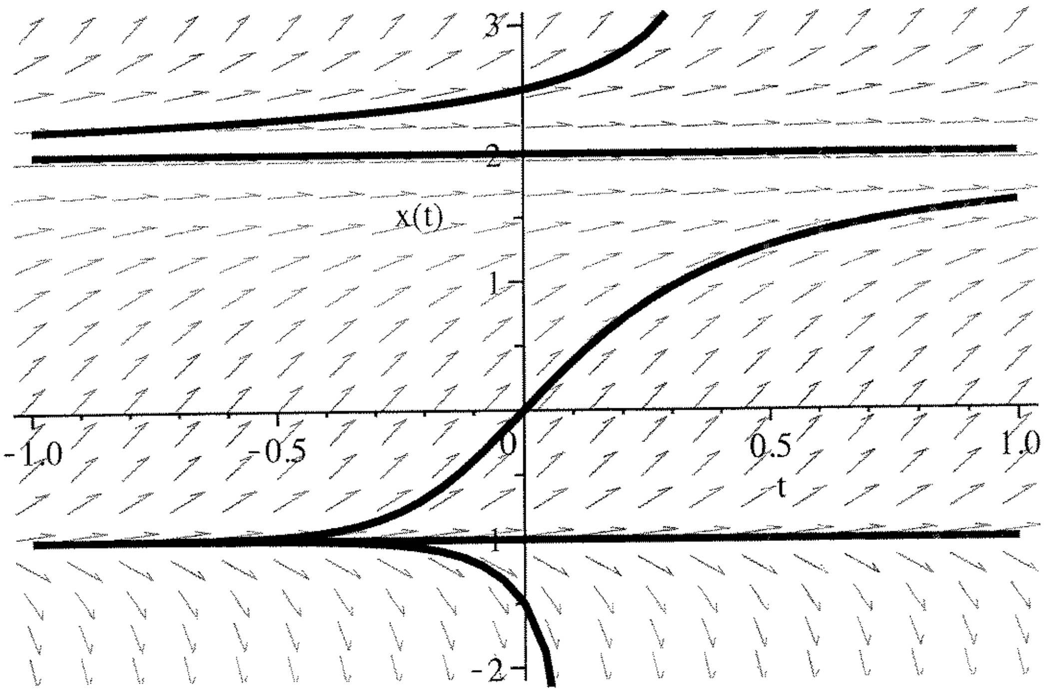
```
> with(DEtools):
  DEplot(diff(x(t),t)=7*x(t)-x(t)^2-10,x(t),t=-1..1,{x(0)=6,x(0)=
  5,x(0)=4,x(0)=2,x(0)=0},x=-5..10,linecolor=black);
```



>

Problem 2.2.13

```
> with(DEtools):
  DEplot(diff(x(t),t)=(x(t)+1)*(x(t)-2)^2,x(t),t=-1..1,{x(0)=2.5,x
  (0)=2,x(0)=0,x(0)=-1,x(0)=-1.5},x=-2..3,linecolor=black);
```



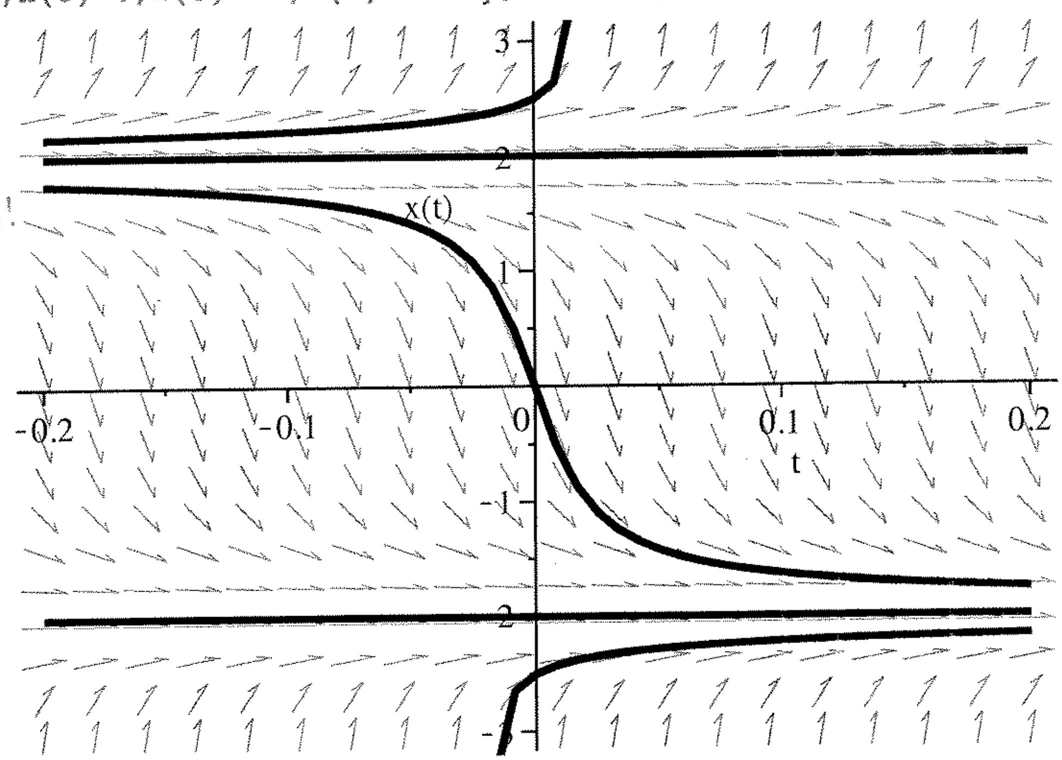
semistable

unstable

Problem 2.2.16

```
> with(DEtools):
  DEplot(diff(x(t),t)=(x(t)^2-4)^3,x(t),t=-0.2..0.2,{x(0)=2.5,x(0)
  =2,x(0)=0,x(0)=-2,x(0)=-2.5},x=-3..3,linecolor=black);
```

note: these solutions blow up very rapidly!



unstable

stable