# MATH 2280-1 Homework # 1 solutions

1.6 We need to check $y_1(x) = e^{-2x}$ and $y_2(x) = xe^{-2x}$ are solutions of the DE

$$y'' + 4y' + 4y = 0.$$

$y_1(x) = e^{-2x}$
$y_1'(x) = -2e^{-2x}$
$y_1''(x) = 4e^{-2x}$

We do have $4e^{-2x} - 8e^{-2x} + 4e^{-2x} = 0$.
$\Rightarrow$ $y_1$ solves DE.

$y_2(x) = xe^{-2x}$
$y_2'(x) = (1-2x)e^{-2x}$
$y_2''(x) = 4(x-1)e^{-2x}$

we do have $4(x-1)e^{-2x} + 4(1-2x)e^{-2x} + 4xe^{-2x} = 0$

$\Rightarrow$ $y_2$ solves DE.

1.1.15 We need to find for which $r$ $y = e^{rx}$ solves the DE $y'' + y' - 2y = 0$. Substituting in LHS for $y$:

$$r^2e^{rx} + re^{rx} - 2e^{rx} = 0 \Leftrightarrow r^2 + r - 2 = 0$$

$$\Leftrightarrow r = \frac{-1 \pm \sqrt{9}}{2}$$

$$\Leftrightarrow \boxed{r = 1 \text{ or } -2.}$$

1.1.16) Same question as in 1.1.15 but with DE $3y'' + 3y' - 4y = 0$
Similarly:

$$3r^2e^{rx} + 3re^{rx} - 4e^{rx} = 0 \Leftrightarrow 3r^2 + 3r - 4 = 0$$

$$\Leftrightarrow \boxed{r = \frac{-3 \pm \sqrt{9+48}}{2 \cdot 3} = \frac{-3 \pm \sqrt{57}}{6}.}$$

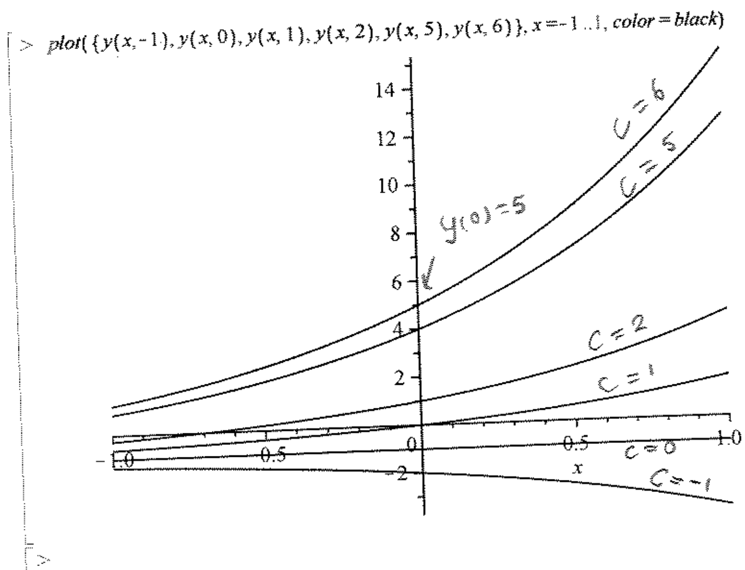
**1.19** let us verify that $y(x) = Ce^{x} - 1$ solves DE $y' = y + 1$.

$y' = Ce^{x} = Ce^{x} - 1 + \; = y + 1$.

To satisfy initial condition $y(0) = 5$ we must have:

$$y(0) = Ce^{0} - 1 = 5 \implies C = 6$$

Several solutions are sketched below.

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**1.27** "The slope of the graph of $g$ at the pt. $(x, y)$ is the sum of $x$ and $y$".

$g$ would then be a solution to:

slope of $y = g(x)$ $\rightarrow$ $\boxed{\dfrac{dy}{dx} = x + y}$.

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**1.32)** "The time rate of change of a population $P$ is proportional to the square root of $P$".

$\boxed{\dfrac{dP}{dt} = k\sqrt{P}}$ represents this situation.

$k$ = proportionality constant.

1.35 : "In a city having a fixed population $P$ of persons, the time rate of change $N$ of those persons who have heard a certain rumor is propotional to # of those not having heard rumor"

$$\boxed{\frac{dN}{dt} = k(P-N)}$$

time rate change of people hearing rumor

prop. constant

# of people not having heard rumor.

1.2.6 Solve the DE

$$\begin{cases} y' = x\sqrt{x^2+9} \\ y(-4)=0 \end{cases} \Rightarrow y(x) = \int x\sqrt{x^2+9}\,dx + C$$

$$= \frac{1}{3}(x^2+9)^{\frac{3}{2}} + C$$

$$0 = y(-4) = \frac{1}{3}(16+9)^{3/2} + C$$

$$\Rightarrow C = -\frac{1}{3}5^3 = -\frac{125}{3}$$

Solution is $\boxed{y(x) = \frac{1}{3}(x^2+9)^{3/2} - \frac{125}{3}}$

1.2.8 Solve the DE

$$\begin{cases} y' = \cos 2x \\ y(0) = 1 \end{cases} \Rightarrow y(x) = \int \cos 2x\,dx + c$$

$$= \frac{1}{2}\sin 2x + c$$

$$1 = y(0) = \frac{1}{2}\sin 2\cdot 0 + c = c$$

$\Rightarrow$ Solution is

$$\boxed{y(x) = \frac{1}{2}\sin 2x + 1}$$

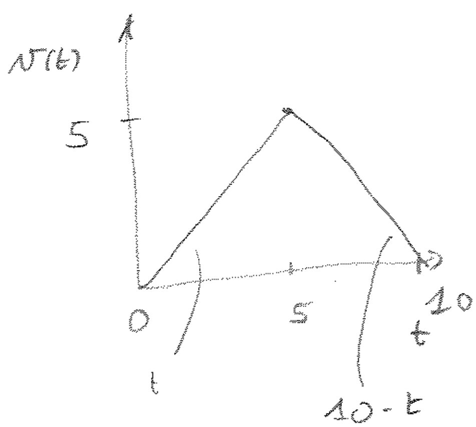
..2.14 We need to find velocity and position of movement if we are given:

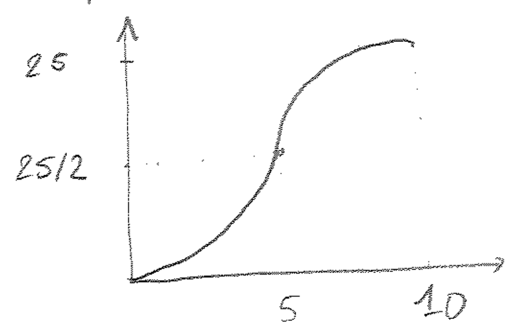
$a(t) = 2t+1$ , $v_0 = -7$, $x_0 = 4$

$$\Rightarrow \boxed{v(t) = \int_0^t 2s+1\, ds + v_0 = t^2+t+v_0 = t^2+t-7}$$

$$\boxed{x(t) = \int_0^t s^2+s-7\, ds + x_0 = \frac{t^3}{3}+\frac{t^2}{2}-7t+4}$$

.2.21 If the particle starts at the origin with velocity profile its position will be:




$$v(t) = \begin{cases} t & \text{if } t \leq 5 \\ 10-t & \text{if } t > 5 \end{cases}$$

$$x(t) = \begin{cases} \frac{1}{2}t^2 & \text{if } t \leq 5 \\ 10t - \frac{1}{2}t^2 - 25 & \text{if } t > 5 \end{cases}$$

..2.32 We have two experiments, thus two trajectories $x_1(t)$ and $x_2(t)$ with velocities $v_1(t)$, $v_2(t)$ and accelerations $a_1(t) = a_2(t) = -\underset{>0}{\underset{\sim}{a}} =$ constant.

first experiment: car skids 15m when $v_1(0) = 50$km/h.

if $T_1$ = time at which car stops then $v_1(T_1) = 0$.

thus:

$$v_1(t) = -at + v_1(0)$$

$$v_1(T_1) = -aT_1 + v_1(0) = 0 \Rightarrow T_1 = \frac{v_1(0)}{a} \quad (1)$$

Moreover:

$$x_1(t) = -\frac{a}{2}t^2 + v_1(0)t \qquad \text{(assuming car breaks at } x_1(0)=0\text{)}$$

$$\Rightarrow \quad x_1(T_1) = 15m = -\frac{a}{2}T_1^2 + v_1(0)\,T_1 \qquad (2)$$

Second experiment: car skids $x_2(T_2)$ distance, where $T_2$ = time it stops.
the answer
thus:

$$v_2(t) = -at + v_2(0)$$

$$v_2(T_2) = -aT_2 + v_2(0) \Rightarrow T_2 = \frac{v_2(0)}{a} \qquad (3)$$

Equating (1) and (3) we get:

$$T_2 = \frac{v_2(0)}{v_1(0)}T_1 = \frac{100km/h}{50km/h}T_1 = 2\,T_1$$

Using (1) in (2):

$$15m = -\frac{v_1(0)\,T_1}{2} + v_1(0)\,T_1$$

$$\Rightarrow \quad T_1 = 2\,\frac{15m}{50km/h} = \frac{3}{5}10^{-3}h$$

$$\Rightarrow \quad T_2 = \frac{6}{5}10^{-3}h$$

and since

$$x_2(T_2) = -\frac{a}{2}T_2^2 + v_2(0)\,T_2$$

where $a = \frac{v_1(0)}{T_1}$ we get:

$$\boxed{x_2(T_2) = -\frac{v_1(0)}{2T_1}T_2^2 + v_2(0)\,T_2}$$

$$= -T_2\,v_1(0) + T_2\,v_2(0)$$

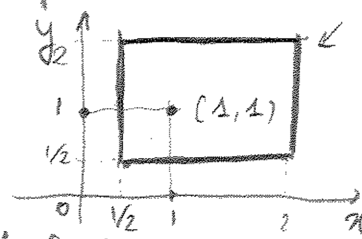
$$= \frac{6}{5}10^{-3}h \times \left(100.10^3 - 50.10^3\right)$$

$$= \boxed{60m}$$

**1.3.12**

$$\begin{cases} \dfrac{dy}{dx} = x \ln y \\ y(1) = 1 \end{cases}$$

Here $f(x,y) = x \ln y$ is continuous and $\dfrac{\partial f}{\partial y}(x,y) = \dfrac{x}{y}$ " " on a rectangle containing $(1,1)$, for example:




$\swarrow R = [1/2, 2] \times [1/2, 2]$

So the hypothesis of theorem 1 hold and we should have a unique sol. in the vicinity of $(1,1)$.

**1.3.13**)

$$\begin{cases} \dfrac{dy}{dx} = y^{1/3} \\ y(0) = 1 \end{cases}$$

Here $f(x,y) = y^{1/3}$ is continuous on a neighborhood of $(0,1)$ and so is $\dfrac{\partial f}{\partial y}(x,y) = \dfrac{1}{3} y^{-2/3}$. Therefore hypothesis of theorem 1 hold for example for rectangle $[-1,1] \times [1/2, 2]$.

$\Rightarrow$ existence and uniqueness guaranteed.

**1.3.14** Same problem as above but with initial cdt° $y(0) = 0$.

This time the two hypothesis are violated:

$f(x,y) = y^{1/3}$ is continuous for $y > 0$, so it is impossible to find a rectangle containing $(0,0)$ in its interior where $f(x,y)$ is continuous

$\dfrac{\partial f}{\partial y}(x,y) = \dfrac{1}{3} y^{-2/3}$ is continuous likewise for $y > 0$, so it is also impossible to construct a rectangle around $(0,0)$ where it is continuous.

$\Rightarrow$ existence and uniqueness not guaranteed.

1.3.23

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> with(DEtools): DEplot(diff(y(x), x) = x^2 + y(x)^2 - 1, y(x), x=-1..3,
  {y(0)=0}, y=-1..2, linecolor=black);
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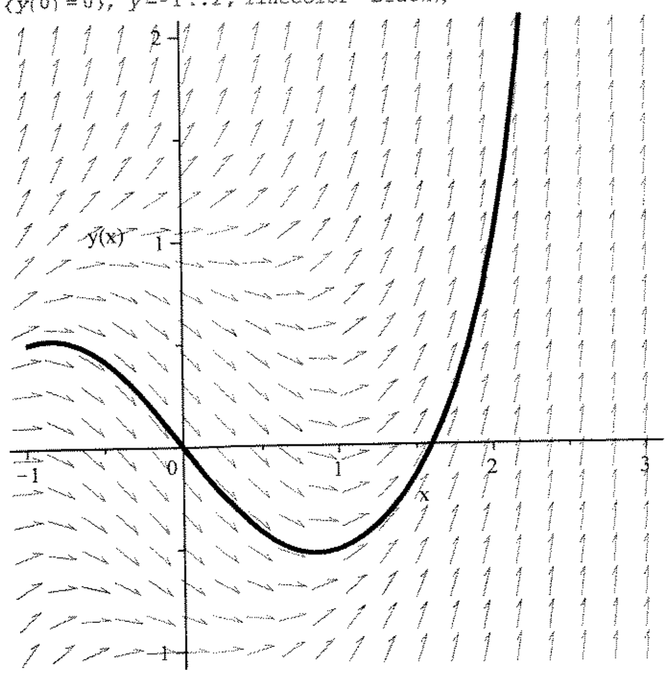


We have $y(2) \approx 1$.

1.3.26

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> with(DEtools): DEplot(diff(P(t), t) = 0.0225 P(t)-0.0003 P(t)^2,
  P(t), t=0..200, {P(0)=25}, P=0..90, linecolor=black);
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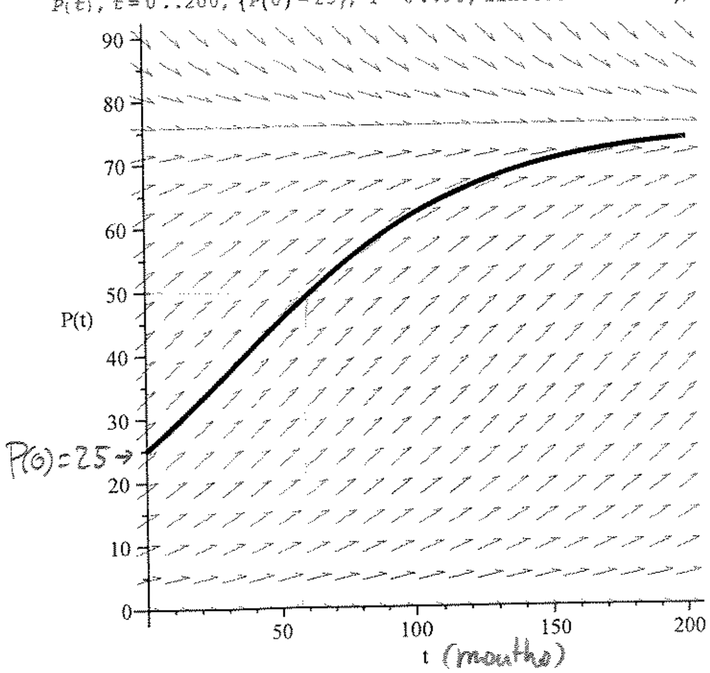



The deer population will double after $\approx$ 60 months.

The limiting population seems to be 75 deer.

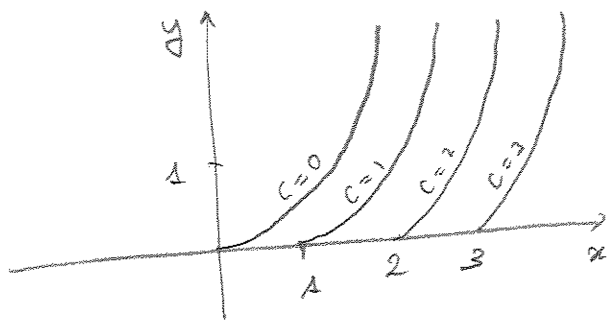
(1.3.27) (a) The function $y(x) = \begin{cases} 0 & \text{for } x \le c \\ (x-c)^2 & \text{for } x > c \end{cases}$ (*) ⑧

is differentiable with:

$$y'(x) = \begin{cases} 0 & \text{for } x \le c \\ 2(x-c) & \text{for } x > c \end{cases}$$

$\Rightarrow$ it satisfies DE $y' = 2\sqrt{y}$. With initial condition $y(0) = 0$, this DE admits infinitely many solutions:

A function of for (*) with $c \geq 0$ is a solution to DE with $y(0)=0$




(b)

The DE $\begin{cases} y' = 2\sqrt{y} \\ y(0) = b \end{cases}$ has:

i) no solution when $b < 0$, since square root is not defined.

ii) a unique solution when $b > 0$, since hypothesis of Theorem 1 (on $\exists$ and !) hold. The solution has form (*) so it is defined over $\mathbb{R}$:

$$y(x) = \begin{cases} 0 & \text{for } x \le -\sqrt{b} \\ x + \sqrt{b} & \text{for } x > -\sqrt{b}. \end{cases}$$

1.4.12: $yy' = x(y^2+1)$. Can be solved using separation of variables:

$$\int \frac{y}{y^2+1}\,dy = \int x\,dx + C$$

$$\frac{1}{2}\ln|y^2+1| = \frac{x^2}{2} + C$$

different constant

$$\ln(y^2+1) = x^2 + C$$

$$\Rightarrow \boxed{y^2 = Ae^{x^2} - 1}, \quad \text{where } A = e^C > 0.$$

1.4.22

$$\begin{cases} \dfrac{dy}{dx} = 4x^3y - y \\ y(1) = -3 \end{cases}$$

Can be solved using separation of variables.

$$\int \frac{dy}{y} = \int (4x^3 - 1)\,dx + C$$

$$\ln|y| = x^4 - x + C$$

Since $y(1) < 0$, it will remain of that sign in a neiborhood of 1:

$$\ln(-y) = x^4 - x + c$$

$$y(x) = -Ae^{x^4 - x} \quad \text{where } A = e^c > 0.$$

now $y(1) = -3 = -Ae^{1-1} \Rightarrow A = 3$

$$\Rightarrow \boxed{y(x) = -3e^{x^4 - x}}$$

1.4.38. Continuously compounded interest.

$A(t) = \#$ of dollars in account

$A(0) = 5000$ (initial deposit)

$$\begin{cases} \dfrac{dA}{dt} = rA \\ A(0) = 5000 \end{cases}$$

and $r = 8\% = 0.08$.

This DE can be solved with sep. of vars:

$A(t) = A(0)e^{rt}$

$A(18) = \$5000\, e^{0.08 \cdot 18} \simeq \$21103.48$.

will be in the account after 18y.

1.4.44 Let $A_0$ be the undissolved amount of sugar at $t=0$.

Then the DE:

$$\begin{cases} \dfrac{dA}{dt} = -kA \\ A(0) = A_0 \end{cases}$$

has solution $A(t) = A_0 e^{-kt}$.

= undissolved sugar at time $t$.

If 25% sugar dissolves after 1 min: $A(1) = A_0 e^{-k} = \frac{3}{4}A_0$

$\Leftrightarrow \quad e^{-k} = \frac{3}{4} \quad \Rightarrow \quad \boxed{k = \ln\left(\frac{4}{3}\right)}$

Let $T$ be the time it takes for half the sugar to dissolve. Then:

$A(T) = \dfrac{A_0}{2} = A_0 e^{-kT} \Rightarrow \quad \boxed{T = \dfrac{\ln 2}{k} \simeq 2.41 \text{ min.}}$

1.4.65 Newton's cooling law:

$\dfrac{dT}{dt} = -k(T-A)$ with $k>0$, and $A$ = ambient temp.

Using sep. of vars we get: $T(t) = A + Ce^{-kt}$, with $C>0$.

We set $t=0$ = noon then:

$$T(0) = 70 + Ce^{-k0} = 80 \Rightarrow C = 10$$

$$T(1) = 70 + 10e^{-k} = 75 \Rightarrow e^{-k} = \frac{1}{2}$$

$$\Rightarrow k = \ln 2.$$

thus:

$$T(t_{death}) = 98.6 = 70 + 10e^{-(\ln 2)t_{death}}$$

$$\Rightarrow -(\ln 2)\, t_{death} = \ln \frac{28.6}{10}$$

$$t_{death} \simeq -1.5160 \text{ hours} \simeq \boxed{10:29 \text{ am}}.$$