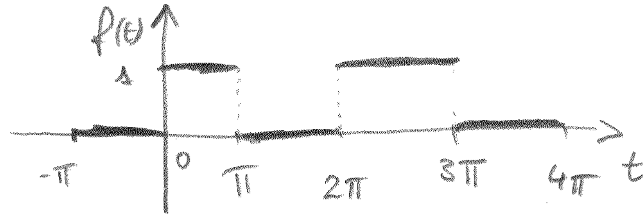
HW 10 Solutions

9.1.13 $f(t) = \begin{cases} 0 & -\pi < t \leq 0 \\ 1 & 0 < t \leq \pi \end{cases}$



Fourier coeff: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_0^{\pi} dt = \boxed{1}$

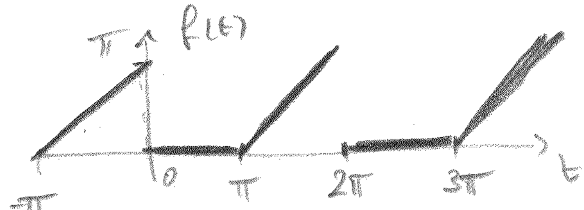
for $n \geq 1$: $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\int_0^{\pi}\cos nt\,dt = \frac{1}{n\pi}\sin nt\Big|_{t=0}^{\pi} \boxed{= 0.}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\int_0^{\pi}\sin nt\,dt$

$= \frac{-1}{n\pi}\cos nt\Big|_{t=0}^{\pi} = \boxed{\frac{(1-(-1)^n)}{n\pi}}$

$\Rightarrow f(t) = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin n\pi = \boxed{\frac{1}{2} + \frac{2}{\pi}\sum_{n\text{ odd}}\frac{\sin n\pi}{n}}$

9.1.19 $f(t) = \begin{cases} \pi + t & \text{if } -\pi < t \leq 0 \\ 0 & \text{if } 0 < t \leq \pi \end{cases}$



Fourier coeff: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{0}(\pi + t)\,dt = \boxed{\frac{\pi}{2}}$ ← book typo

for $n \geq 1$ $a_n = \frac{1}{\pi}\int_{-\pi}^{0}(\pi + t)\cos nt\,dt \overset{IBP}{=} \frac{1}{\pi}(\pi + t)\frac{\sin nt}{n}\Big|_{-\pi}^{0} - \frac{1}{n\pi}\int_{-\pi}^{0}\sin nt\,dt$

$= \frac{+1}{n^2\pi}\cos nt\Big|_{-\pi}^{0} = \boxed{\frac{1-(-1)^n}{n^2\pi}}$

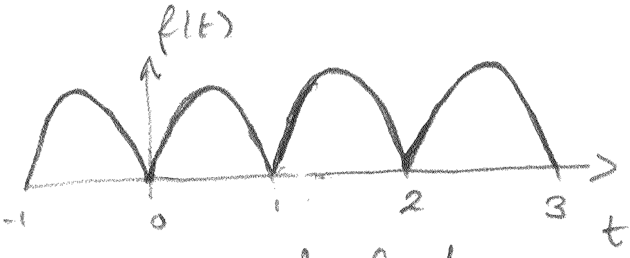
$b_n = \frac{1}{\pi}\int_{-\pi}^{0}(\pi + t)\sin nt\,dt = -\frac{1}{\pi n}(\pi + t)\cos nt\Big|_{-\pi}^{0} + \frac{1}{n\pi}\int_{-\pi}^{0}\cos nt\,dt$

$= -\frac{1}{n} + \cancel{\frac{1}{n^2\pi}\sin nt\Big|_{t=0}^{\pi}} = \boxed{-\frac{1}{n}}$

$\Rightarrow \boxed{f(t) = \frac{\pi}{4} + \frac{2}{\pi}\sum_{n\text{ odd}}\frac{\cos nt}{n^2} - \sum_{n=1}^{\infty}\frac{\sin nt}{n}}$

(2)

9.2.12 $\quad f(t) = \sin \pi t, \quad 0 < t < 1$



1- periodic function $\Rightarrow L = \frac{1}{2}$

f even $\Rightarrow b_n = 0, n \geq 1.$

$$a_0 = 2\int_0^1 \sin \pi t \, dt = -2 \left. \frac{\cos \pi t}{\pi} \right|_0^1 = \boxed{\frac{4}{\pi}}$$

$$a_n = 2\int_0^1 \sin \pi t \cos 2\pi n t \, dt = \int_0^1 \sin((2\pi n + \pi)t) + \sin(\pi - 2\pi n)t \, d$$

using trig formula

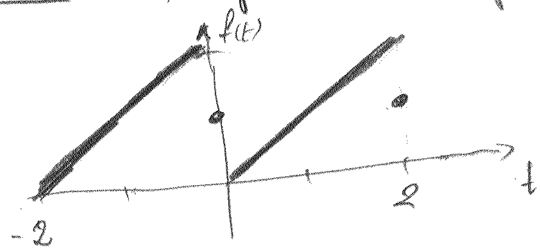
$2\sin a \cos b = \sin(a+b) + \sin(a-b)$

$$= \left. \frac{-\cos(\pi(2n+1)t)}{\pi(2n+1)} - \frac{\cos(\pi(1-2n)t)}{\pi(1-2n)} \right|_{t=0}^{1}$$

$$= -\frac{\overset{(-1)}{\cancel{\cos}(\pi(2n+1))}}{\pi(2n+1)} - \frac{\overset{(-1)}{\cancel{\cos(\pi(1-2n))}}}{\pi(1-2n)} + \frac{1}{\pi(2n+1)} + \frac{1}{\pi(1-2n)}$$

$$= \boxed{\frac{4}{\pi(1-4n^2)}}$$

9.2.17 (a) $f(t) = t$ for $0 < t < 2$ and f is 2-periodic.



Fourier coeff:

$$a_0 = \int_0^2 t \, dt = \left. \frac{t^2}{2} \right|_0^2 = 2$$

$$n \geq 1, \quad a_n = \int_0^2 t \cos n\pi t \, dt = \left. \cancel{\frac{t \sin n\pi t}{n\pi}} \right|_0^2 - \int_0^2 \frac{\sin n\pi t}{n\pi}$$

$$= \left. \frac{\cos n\pi t}{(n\pi)^2} \right|_0^2 = 0.$$

$$n \geq \quad b_n = \int_0^2 t \sin n\pi t \, dt = -t \left. \frac{\cos n\pi t}{n\pi} \right|_0^2 + \int_0^2 \frac{\cos n\pi t}{\pi n}$$

$$= -\frac{2}{n\pi} + \left. \cancel{\frac{\sin n\pi t}{(\pi n)^2}} \right|_0^2$$

$$\Rightarrow \boxed{f(t) = 1 - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi t}{n}}$$

9.2.17 (cont'd)

(b) We have: $f\left(\frac{1}{2}\right) = \frac{1}{2} = 1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n} = 1 - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)}$

since $\sin\frac{n\pi}{2} = \begin{cases} 0 & \text{if } n \text{ even} \\ (-1)^{\frac{(n-1)}{2}} & \text{if } n \text{ odd} \end{cases}$

thus: $-\frac{1}{2} = -\frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\cdots\right)$

$\Rightarrow \boxed{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\cdots = \frac{\pi}{4}}$

9.3.1 $f(t) = 1$, for $0 < t \leq \pi$.

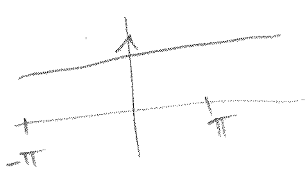
odd extension:

$\boxed{b_n} = \frac{2}{\pi}\int_0^{\pi}\sin nt\,dt = \frac{2}{\pi}\left(-\frac{\cos nt}{n}\right)\Big|_{t=0}^{\pi} = -\frac{2((-1)^n - 1)}{n\pi}$

$= \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

$\Rightarrow \boxed{f_o(t) = \sum_{k=0}^{\infty}\frac{4}{(2k+1)\pi}\sin((2k+1)t)}$

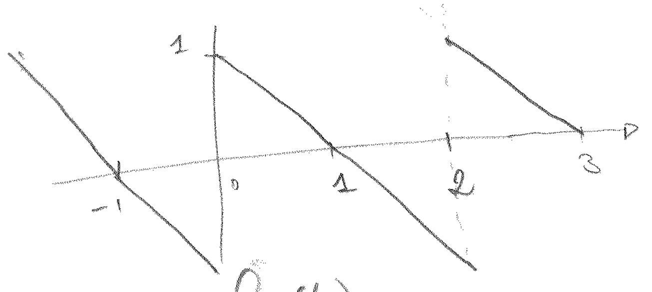
even extension



$\boxed{f(t) = 1}$ = cosine series!

9.3.2 $f(t) = 1 - t$, $0 < t < 1$. Sine series for odd extension.

odd extension.



$f_o(t)$

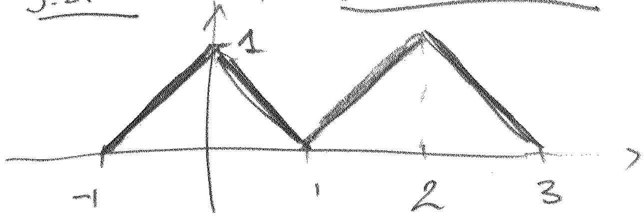
$\boxed{b_n} = 2\int_0^1 (1-t)\sin n\pi t\,dt$

$= -\frac{2}{n\pi}(1-t)\cos n\pi t\Big|_0^1 - \frac{2}{n\pi}\int_0^1\cos n\pi t\,dt$

$= \frac{2}{n\pi} - \underbrace{\frac{2}{(n\pi)^2}\sin n\pi t\Big|_0^1}_{0}$

$\boxed{f_o(t) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin n\pi t}$

9.3.2 Cont'd even extension



$f_e(t)$

$$\boxed{f_e(t) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{((2k+1)\pi)^2}}$$

Cosine series for even extension

$n \geq 1$, $a_n = 2\int_0^1 (1-t)\cos n\pi t \, dt$

$$= \underbrace{\frac{2}{n\pi}(1-t)\sin n\pi t \Big|_0^1}_{=0} + \frac{2}{n\pi}\int_0^1 \sin n\pi t \, dt$$

$$= -\frac{2}{(n\pi)^2}\cos n\pi t \Big|_0^1 = \frac{-2}{(n\pi)^2}\left((-1)^n - 1\right)$$

$$= \begin{cases} \frac{2}{(n\pi)^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$a_0 = 2\int_0^1 (1-t)\, dt = 1$$

9.3.12

$$\begin{cases} x'' - 4x = 1 \\ x(0) = x(\pi) = 0 \end{cases}$$

We look for x of the form:

$$x(t) = \sum_{n=1} b_n \sin nt$$

since we have $\sin nt \Big|_{t=0} = 0$

and $\sin nt \Big|_{t=\pi} = 0$

$\Rightarrow$ x satisfies B.C. automatically.

The sine coeff (or Fourier coeff of the odd extension) of $f(t) = 1$ are:

$$B_n = \frac{2}{\pi}\int_0^{\pi} 1 \cdot \sin nt \, dt = -\frac{2}{n\pi}\cos nt \Big|_0^{\pi} = -\frac{2}{n\pi}\left((-1)^n - 1\right)$$

$$= \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

Thus:

$$x'' - 4x = 1 = \sum_{n \text{ odd}} \frac{4}{n\pi}\sin nt$$

$$\| $$

$$\sum_{n=1}^{\infty} b_n(-n^2 - 4)\sin nt$$

$$\Rightarrow \quad b_n = \begin{cases} \frac{-4}{n\pi(n^2+4)} & n \text{ odd} \\ 0 \text{ if} & n \text{ even} \end{cases} \Rightarrow \boxed{x(t) = \sum_{k=0}^{\infty} \frac{-4 \sin (2k+1)t}{(2k+1)\pi((2k+1)^2+4)}}$$

9.4.1 $x'' + 5x = F(t)$, where $F(t) = \begin{cases} 3 & \text{for } 0 < t \leq \pi \\ -3 & \text{for } \pi < t \leq 2\pi \end{cases}$

and $F(t)$ is 2π periodic

$F(t)$ is up to a constant multiplicative factor the same function we studied in 9.3.1 (odd extension) then:

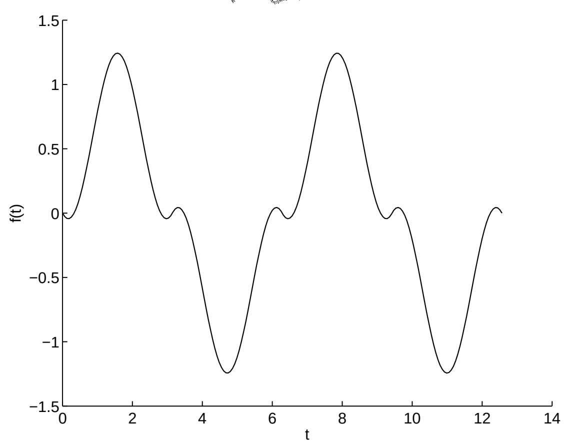
$$\boxed{F(t) = \frac{12}{\pi} \sum_{n \text{ odd}} \frac{\sin nt}{n}}$$

To find a steady periodic solution to the DE we take then:

$$x(t) = \sum_{n=1}^{\infty} b_n \sin nt$$

Thus:

$$\underset{\displaystyle \sum_{n=1}^{\infty} b_n (5-n^2) \sin nt}{\underset{\shortparallel}{x'' + 5x}} = \underset{\displaystyle \sum_{n \text{ odd}} \frac{12}{\pi} \frac{\sin nt}{n}}{\underset{\shortparallel}{F(t)}} \Rightarrow b_n = \begin{cases} \dfrac{12}{n\pi(5-n^2)} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$



9.5.3

$$\begin{cases} u_t = 2u_{xx} \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 5 \sin \pi x - \frac{1}{5} \sin 3\pi x \end{cases}$$

Let $u(x,t) = X(x)T(t)$

$\Rightarrow XT' = 2X''T$

$\Rightarrow \dfrac{T'}{2T} = \dfrac{X''}{X} = -\lambda = \text{const.}$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(1) = 0 \end{cases} \Rightarrow \begin{array}{l} X_n(x) = \sin n\pi x \\ \lambda_n = (n\pi)^2 \end{array}$$

$$T_n' + 2\lambda_n T_n = 0 \Rightarrow T_n(t) = e^{-2(n\pi)^2 t}$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} b_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} b_n \sin n\pi x \, e^{-2(n\pi)^2 t}$$

Using the initial conditions:

$$u(0,t) = \sum_{n=1}^{\infty} b_n \sin n\pi x = 5 \sin \pi x - \frac{1}{5} \sin 3\pi x$$

$$\Rightarrow \quad b_1 = 5, \; b_3 = -\frac{1}{5}, \; b_n = 0 \text{ for } n \in \mathbb{N} \setminus \{1, 3\}.$$

$$\Rightarrow \boxed{u(x,t) = 5 \sin \pi x \, e^{-2\pi^2 t} - \frac{1}{5} \sin 3\pi x \, e^{-2(3\pi)^2 t}}$$