Math 2280-2, Spring 2008
Final exam correction

May 5, 2008

Point schedule

Problem 1. (a) 10, (b) 10; tot=20

Problem 2. (a) 5, (b) 5, (c), 10, (d) 10, (e) 10; tot=40

Problem 3. (a) 5, (b) 10, (c) 5, (d) 10; tot=30

Problem 4. (a) 10, (b) 10, (c) 10, (d) 10; tot=40

Problem 5. (a) 8, (b) 8, (c) 8, (d) 8, (e) 8; tot=40

Problem 6. (a) 15, (b) 5, (c) 5, (d) 5, tot=30

Total: 200 points possible, but grade was over 170.

Correction

Math 2280-2, Final Exam
April 29 2008

April 29, 2008

- **NAME:** ....
- **STUDENT ID:** ....
- No books or notes allowed
- Only scientific calculators without DE solving capability are allowed.
  However a calculator is NOT required for completing this exam.
- Two hour time limit
- Before starting a problem, READ every question.
  Some questions have a simple answer that does not require many calculations.
- **Total:** 170/200 points

Problem 1 (20 pts) Consider the initial value problem

$$\begin{cases} x^{(4)} - x = 0 \\ x(0) = 1,\ x'(0) = x''(0) = x'''(0) = 0 \end{cases}$$

Solve this problem with the following methods
(**hint:** $x(t) = (\cos(t) + \cosh(t))/2$ and $s^4 - 1 = (s^2 - 1)(s^2 + 1)$)

(a) Laplace transform
(**hint:** see the end of exam for a table of Laplace transforms and properties of hyperbolic trigonometric functions.)

(b) Characteristic polynomial.

(a) $\overset{\mathcal{L}}{\rightsquigarrow}$ $s^4 X(s) - X(s) = s^3$

$$X(s) = \frac{s^3}{s^4-1} = \frac{s^3}{(s^2-1)(s^2+1)} = \frac{s^3}{(s-1)(s+1)(s^2+1)}$$

$$= \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s^2+1} + \frac{Ds}{s^2+1}$$

mult. by denominator:

$\Rightarrow s^3 = A(s+1)(s^2+1) + B(s-1)(s^2+1) + C(s^2-1) + Ds(s^2-1)$

Equating terms we get:

in s^3: $1 = A + B + D$ (1)

in s^2: $0 = A - B + C$ (2)

in s: $0 = A + B - D$ (3)

in 1: $0 = A - B - C$ (4)

$(2)+(3) \Rightarrow A = B$

$(1)-(3) \Rightarrow D = \frac{1}{2}$

thus $(3) \Rightarrow A + B = \frac{1}{2}$

$\Rightarrow A = B = \frac{1}{4}$

$$\Rightarrow X(s) = \frac{1}{4}\left(\frac{1}{s-1} + \frac{1}{s+1}\right) + \frac{1}{2}\frac{s}{s^2+1}$$

$\overset{\mathcal{L}^{-1}}{\rightsquigarrow}$ $\boxed{x(t) = \frac{1}{4}(e^{-t} + e^{t}) + \frac{1}{2}\cos t}$

$$\boxed{= \frac{1}{2}\cosh t + \frac{1}{2}\cos t}$$

(b) The characteristic polyn of eq is $r^4 - 1 = 0$

$\Rightarrow r = \begin{cases} \pm 1 \\ \pm i \end{cases}$ $\qquad (r^2-1)(r^2+1)$

Thus
$$
\begin{aligned}
x(t) &= a\cosh t + b\sinh t + c\cos t + d\sin t \\
x'(t) &= a\sinh t + b\cosh t - c\sin t + d\cos t \\
x''(t) &= a\cosh t + b\sinh t - c\cos t - d\sin t \\
x'''(t) &= a\sinh t + b\cosh t + c\sin t - d\cos t
\end{aligned}
\qquad
\left\{
\begin{aligned}
x(0) &= a+c = 1 \\
x'(0) &= b+d = 0 \\
x''(0) &= a-c = 0 \\
x'''(0) &= b-d = 0
\end{aligned}
\right.
$$

$\Rightarrow \quad a = c = \frac{1}{2} \quad b = d = 0.$

$\Rightarrow \boxed{x(t) = \frac{1}{2}\cos t + \frac{1}{2}\cosh t}$

Problem 2 (40 pts) Consider the two populations system

$$\begin{cases} x' = 3x - x^2 - xy \\ y' = y - xy \end{cases}$$

(a) How does each population behave in the absence of the other one (logistic, exponential growth or decrease)?

x- population is logistic
y- population has natural (exp.) growth

(b) Describe the nature of the interaction between the x−population and the y−population (competition, cooperation, predation)

The larger the y- pop the smaller the rate at which x increases
——— x-pop ——— y increases

$\Rightarrow$ we have two species in competition

(c) Find the three critical points of the population (**hint:** one of them is $(1, 2)^T$).

We need to solve:

$$\begin{cases} 3x - x^2 - xy = 0 \\ y - xy = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x(3-x-y) = 0 & (1) \\ y(1-x) = 0 & (2) \end{cases}$$

$$\begin{cases} x = 0 \overset{(2)}{\Rightarrow} y = 0 \\ y = 0 \overset{(1)}{\Rightarrow} x = 0 \text{ or } x = 3 \\ x = 1 \overset{(1)}{\Rightarrow} y = 2 \end{cases}$$

thus crit pts are: $(0,0)$ $(3,0)$ & $(1,2$

(d) Classify the type and stability of each critical point. (**note: No sketch is necessary**)

The Jacobian of the system is:

$$J(x,y) = \begin{bmatrix} 3-2x-y & -x \\ -y & 1-x \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \rightsquigarrow \underline{\text{unstable node}} \quad (\lambda = 3, 1)$$

$$J(3,0) = \begin{bmatrix} -3 & -3 \\ 0 & -2 \end{bmatrix}$$

$\lambda = -2, -3$

$\rightsquigarrow$ <u>stable node</u>

$$J(1,2) = \begin{bmatrix} -1 & -1 \\ -2 & 0 \end{bmatrix} \qquad p(\lambda) = (-1-\lambda)(-\lambda) - 2$$
$$= \lambda^2 + \lambda - 2$$
$$\lambda = \frac{-1 \pm \sqrt{1+8}}{2} = \begin{cases} 1 \\ -2 \end{cases}$$

$\longrightarrow$ unstable saddle pt

eigenvectors:

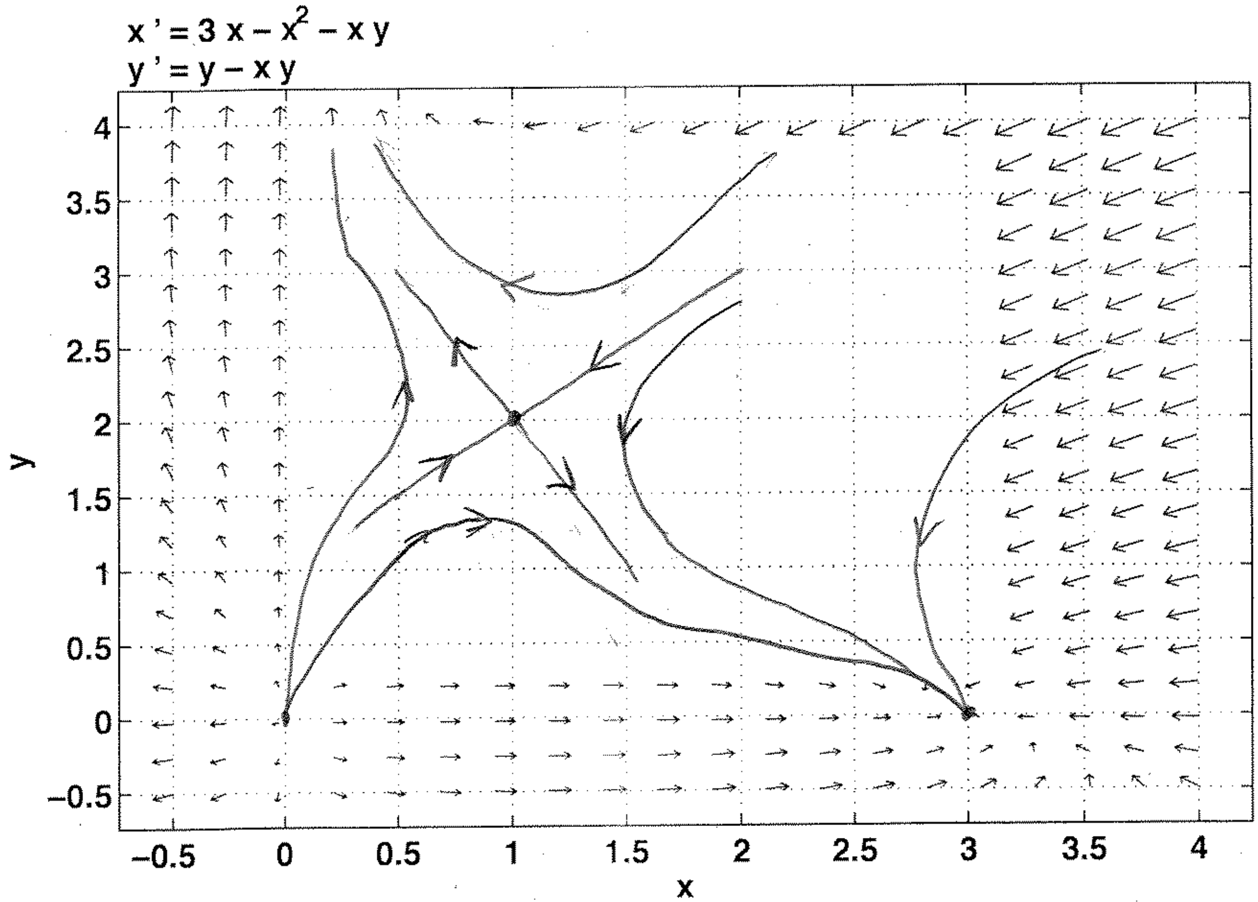
- $\lambda = 1$: find $\underline{v}_1$ s.t. $(A - I)\,\underline{v}_1 = \underline{0}$

$$\begin{bmatrix} -2 & -1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \boxed{\underline{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

- $\lambda = -2$: find $\underline{v}_2$ s.t. $(A + 2I)\,\underline{v}_2 = 0$

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

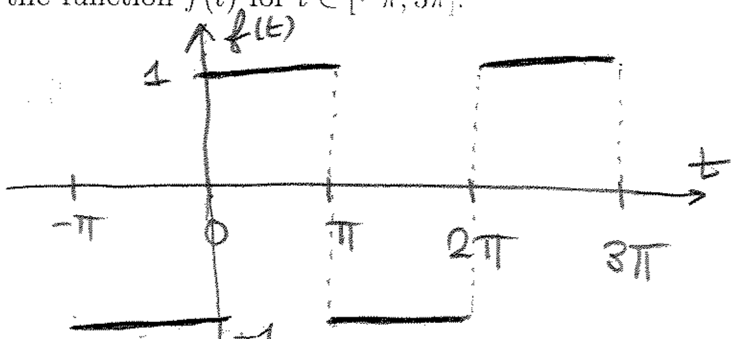
(e) Complete the phase portrait below around the critical point $(1, 2)^T$, and add a few representative trajectories, according to your classification in (d). (**note:** if the point is a saddle point, carefully plot the axis of the hyperbolic trajectories, if the point is a spiral, find the orientation by e.g. computing the tangent field at a couple of points)



Problem 3 (30 pts) Let $f(t)$ be the 2π periodic function defined below

$$f(t) = \begin{cases} 1 & \text{for } 0 < t \le \pi, \\ -1 & \text{for } -\pi < t \le 0. \end{cases}$$

(a) Sketch the function $f(t)$ for $t \in [-\pi, 3\pi]$.



(b) Show that the Fourier series of $f(t)$ is

$$f(t) = \sum_{n \text{ odd}} \frac{4}{n\pi} \sin nt.$$

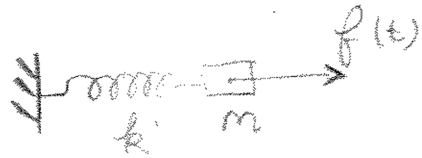
$f(t)$ is odd $\Rightarrow a_n = 0$ for $n \geq 0$

$$b_n = \frac{2}{\pi} \int_0^{\pi} 1 \sin nt \, dt = \left(\frac{2}{\pi}\right) \left(-\frac{\cos nt}{n}\right) \Bigg|_{t=0}^{\pi}$$

$$= \frac{2}{\pi n} \left(1 - (-1)^n\right) = \begin{cases} \frac{4}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

thus $\boxed{f(t) = \sum_{n \text{ odd}} \frac{4}{n\pi} \sin nt = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin (2k+1) t}$

(c) Consider the mass spring system depicted below with $m = 1$ and $k = 9$.



Without solving the DE determine if resonance occurs.

The natural freq of this system is $\omega_0 = \sqrt{\frac{k}{m}} = 3$

Since the forcing term contains $\sin 3t$, the mass spring system will enter resonance.

(d) Use the Fourier series for $f(t)$ at an appropriate value of t to evaluate the series,

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots$$

$$1 = f\left(\frac{\pi}{2}\right) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi/2)}{(2k+1)}$$

now notice $\sin\left((2k+1)\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2} + k\pi\right) = (-1)^k$

thus,

$$\boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}}$$

Problem 4 (40 pts) Consider the first order system of differential equations $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}.$$

note: The matrix A has the nice property that $A^{-1} = A$ (A is "idempotent"). This can greatly simplify your calculations in (d).

(a) Find the eigenvalues and eigenvectors of A. **hint:** the eigenvalues are -1 and 1.

$$p(\lambda) = \begin{vmatrix} -3-\lambda & -2 \\ 4 & 3-\lambda \end{vmatrix} = (-3-\lambda)(3-\lambda) + 8$$

$$= \lambda^2 - 9 + 8 = \lambda^2 - 1 \Rightarrow \lambda = \pm 1$$

eigenvectors:

- $\lambda_1 = 1$ find v_1 s.t. $(A - I)v_1 = 0$

$$\begin{bmatrix} -4 & -2 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \boxed{v_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}}$$

- $\lambda_2 = -1$ find v_2 s.t. $(A + I)v_2 = 0$

$$\begin{bmatrix} -2 & -2 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(b) Find a fundamental matrix solution $\Phi(t)$ to the system $\mathbf{x}' = A\mathbf{x}$.

$$\boxed{\Phi(t)} = [e^t v_1, e^{-t} v_2]$$

$$\boxed{= \begin{bmatrix} -e^t & e^{-t} \\ 2e^t & -e^{-t} \end{bmatrix}}$$

$$\Phi(0) = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \Rightarrow \Phi(0)^{-1} = (-1)\begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

(c) Find the matrix exponential e^{At}.
What is the solution to $\mathbf{x}' = A\mathbf{x}$, with $\mathbf{x}(0) = \mathbf{x}_0$?

$$e^{At}\underline{x}_0 \text{ solves } \begin{cases} \underline{x}' = A\underline{x} \\ \underline{x}(0) = \underline{x}_0 \end{cases}$$

$$e^{At} = \Phi(t)\Phi(0)^{-1} = \begin{bmatrix} -e^{t} & e^{-t} \\ 2e^{t} & -e^{-t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -e^{t}+2e^{-t} & -e^{t}+e^{-t} \\ 2e^{t}-2e^{-t} & 2e^{t}-e^{-t} \end{bmatrix}}$$

(d) Find a particular solution $\mathbf{x}_p(t)$ to $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, where $\mathbf{f}(t) = (1, t)^T$.
hint: Use $\mathbf{x}_p(t) = t\mathbf{a} + \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.

$$\left.\begin{array}{l} \underline{x}_p' = A\underline{x}_p + \begin{pmatrix} 1 \\ t \end{pmatrix} \\ \underline{x}_p = t\underline{a} + \underline{b} \end{array}\right\} \Rightarrow \underbrace{\underline{a}}_{=\underline{x}_p'} = A(t\underline{a} + \underline{b}) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} A\underline{a} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ \text{and } A\underline{b} = \underline{a} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$$

hint

We get: $\underline{a} = A^{-1}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = A\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

$$\underline{b} = A^{-1}\left[\begin{pmatrix} 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = A\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

$$\Rightarrow \boxed{\underline{x}_p = t\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 3 \\ -5 \end{pmatrix} = \begin{pmatrix} 2t + 3 \\ -3t - 5 \end{pmatrix}}$$

(c) Find the matrix exponential e^{At}.
What is the solution to $\mathbf{x}' = A\mathbf{x}$, with $\mathbf{x}(0) = \mathbf{x}_0$?

$e^{At}\underline{x}_0$ solves $\begin{cases} \underline{x}' = A\underline{x} \\ \underline{x}(0) = \underline{x}_0 \end{cases}$

$$e^{At} = \Phi(t)\Phi(0)^{-1} = \frac{1}{2}\begin{bmatrix} -e^{t} & e^{-t} \\ 2e^{t} & -e^{-t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$\boxed{= \frac{1}{2}\begin{bmatrix} -e^{t}+2e^{-t} & -e^{t}+e^{-t} \\ 2e^{t}-2e^{-t} & 2e^{t}-e^{-t} \end{bmatrix}}$$

(d) Find a particular solution $\mathbf{x}_p(t)$ to $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, where $\mathbf{f}(t) = (1, t)^T$.
hint: Use $\mathbf{x}_p(t) = t\mathbf{a} + \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.

$$\left.\begin{array}{l} \underline{x}_p' = A\underline{x}_p + \begin{pmatrix} 1 \\ t \end{pmatrix} \\ \underline{x}_p = t\underline{a} + \underline{b} \end{array}\right\} \Rightarrow \underbrace{\underline{a}}_{=\underline{x}_p'} = A(t\underline{a} + \underline{b}) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} A\underline{a} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ \text{and } A\underline{b} = \underline{a} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$$

hint

We get: $\underline{a} = A^{-1}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = A\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

$$\underline{b} = A^{-1}\left[\begin{pmatrix} 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = A\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

$$\Rightarrow \boxed{\underline{x}_p = t\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 3 \\ -5 \end{pmatrix} = \begin{pmatrix} 2t+3 \\ -3t-5 \end{pmatrix}}$$

Problem 5 (40 pts) The displacement $x(t)$ around equilibrium of a certain physical system is governed by the DE

$$x'' + 4x - x^2 = 0.$$

(a) Write the DE in first-order system form (**hint:** let $y = x'$).

$$\begin{cases} x' = y \\ y' = -4x + x^2 \end{cases}$$

(b) Find the two critical points of the first order system.

$$\begin{cases} y = 0 \\ -4x + x^2 = 0 \end{cases}$$

$$-4x + x^2 = x(-4+x)$$

thus the two critical points are $\boxed{(0,0) \text{ and } (4,0)}$

(c) Find the linearized systems at each of the critical points.

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ -4+2x & 0 \end{bmatrix}$$

At $(0,0)$: $\quad x = u, \quad y = v$

$$\boxed{\begin{pmatrix} u \\ v \end{pmatrix}' = J(0,0)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}}$$

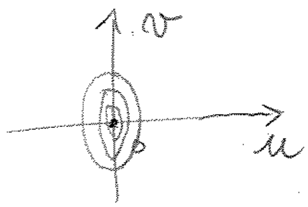
At $(4,0)$: $\quad x = 4+u, \quad y = v$

$$\boxed{\begin{pmatrix} u \\ v \end{pmatrix}' = J(4,0)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}.}$$

(d) Classify the type and stability of the origin for each linearized system, including an accurate **sketch**. (**note:** if the point is a saddle point, carefully plot the axis of the hyperbolic trajectories, if the point is a spiral, find the orientation by e.g. computing the tangent field at a couple of points)

For $(0,0)$ $p(\lambda) = \lambda^2 + 4 \Rightarrow \lambda = \pm 2i$

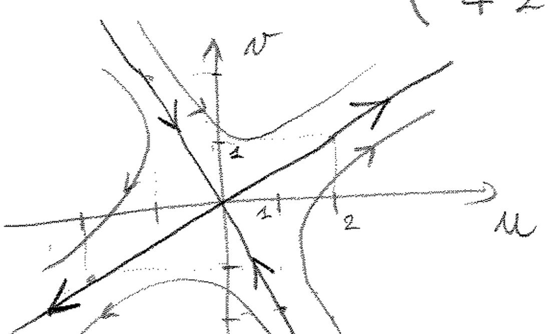
$\Rightarrow$ the origin is a stable center for the lin. problem.



For $(4,0)$: $p(\lambda) = \lambda^2 - 4 \Rightarrow \lambda = \pm 2 \Rightarrow$ unstable saddle pt.

eigenvectors $\lambda_1 = 2$: $\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda_2 = -2$ $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$



(e) What can you conclude about the type and stability of the critical points for the non-linear system, from this linearization study?

$(0,0)$: Since we have 2 purely imaginary eigenvalues, stability is inconclusive from linearization.

$(4,0)$: this point is guaranteed to be an unstable saddle pt in the nonlinear problem as well.

Problem 6 (30 pts) Consider a metallic rod of length π with both ends (and its lateral sides) insulated. The temperature $u(x,t)$ at a point x of the rod at time t satisfies the 1D heat equation:

$$\begin{cases} u_t = k u_{xx}, & \text{for } t > 0 \text{ and } 0 < x < \pi, \quad (1) \\ u_x(0,t) = u_x(\pi,t) = 0, & \text{for } t > 0, \quad (2) \\ u(x,0) = f(x), & \text{for } 0 < x < \pi, \quad (3) \end{cases}$$

where k is the heat conductivity and the initial temperature distribution $f(x)$ is

$$f(x) = 10 - 2\cos(2x) + \frac{1}{5}\cos(30x).$$

(a) Find the temperature distribution $u(x,t)$ of the rod for $t > 0$.

$u(x,t) = X(x)T(t) \overset{(1)}{\Rightarrow} XT' = k X''T \Rightarrow \frac{T'}{kT} = \frac{X''}{X} = -\lambda = \text{const}.$

$\begin{cases} X'' + \lambda X = 0 \\ T' + \lambda k T = 0 \end{cases}$ $\quad \begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(\pi) = 0 \Leftarrow \text{from (2)} \end{cases}$

since

$0 = u_x(0,t) = X'(0)T(t)$

$\Leftrightarrow X'(0)$.

$\lambda = 0$: $\quad X = ax + b$

$X'(0) = a = 0$

$\Rightarrow X(0) = b = \text{const}$

$\lambda < 0$: gives no interesting solutions.

$\lambda = \alpha^2 > 0$: $\quad X(t) = a\cos \alpha x + b \sin \alpha x$

$X'(t) = -a\alpha \sin \alpha x + b\alpha \cos \alpha x$

$X'(0) = b\alpha = 0 \Rightarrow b = 0$

$X'(\pi) = -a\alpha \sin \alpha\pi = 0$

$\Rightarrow \alpha\pi = n\pi \Rightarrow \alpha_n = n$ and $\lambda_n = n^2$

Putting both cases together: $\quad X_n(x) = \cos nx$ for $n \geq 0$.

Now: $T_n' + n^2 k T_n = 0$

$\Rightarrow T_n(t) = A e^{-n^2 kt}$

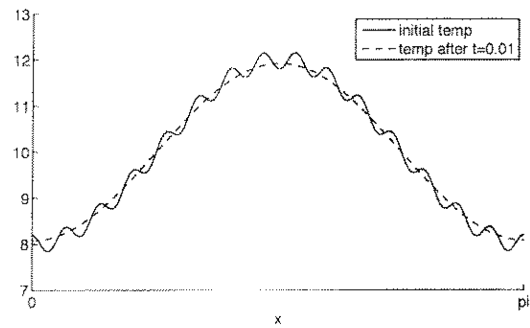
thus : $u(x,t) = \sum_{n=0}^{\infty} a_n X_n(x) T_n(t)$

$\overset{(3)}{\Rightarrow} \boxed{u(x,t) = 10 - 2e^{-4kt}\cos 2t + \frac{1}{5}e^{-900kt}\cos 30t}.$

(b) Explain why the rod's temperature can be for all practical purposes approximated by

$$u(x,t) \approx 10 - 2e^{-4kt}\cos(2x), \quad \text{for } t > 0. \tag{1}$$

To illustrate this, we plot below both $u(x,0) = f(x)$ (solid line) and $u(x, 1/100)$ (dashed line) for $k = 1$. The temperature distribution at $t = 1/100$ is indistinguishable at the scale of this plot from its approximation (1).



What happens is that $e^{-4kt} \gg e^{-900kt}$ so the last term in the sum for $u(x,t)$ becomes very small for $t>0$.

$\Rightarrow u(x,t) \approx 10 - 2e^{-4kt}\cos(2x)$

only first two terms

(c) Based on approximation (1) and with $k = 1$, estimate the time T_0 it takes for the maximum temperature of the rod to reach 11 (temperature units).

The max temp is reached at the center of the rod ($x = \frac{\pi}{2} \Rightarrow \cos 2\frac{\pi}{2} = -$
Thus we must find T_0 s.t.

$11 = 10 + 2e^{-4T_0} \Rightarrow e^{-4T_0} = \frac{1}{2}$

$\Rightarrow T_0 = \frac{\ln 2}{4} \approx 0.17$, (time unit

(d) What is the limiting temperature when $t \to \infty$?

From (1) we get: $\lim_{t\to\infty} u(x,t) = 10$.

Table 1: Table of Laplace transforms

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f^{(3)}(t)$	$s^3F(s) - s^2f(0) - sf'(0) - f''(0)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
$\cos kt$	$\dfrac{s}{s^2+k^2}$
$\sin kt$	$\dfrac{k}{s^2+k^2}$
$\cosh kt$	$\dfrac{s}{s^2-k^2}$
$\sinh kt$	$\dfrac{k}{s^2-k^2}$
e^{at}	$\dfrac{1}{s-a}$

Table 2: Properties of hyperbolic trigonometric functions

$$\begin{aligned} \cosh x &= \frac{e^x + e^{-x}}{2}, & \sinh x &= \frac{e^x - e^{-x}}{2}, \\ (\cosh x)' &= \sinh x, & (\sinh x)' &= \cosh x, \\ \cosh 0 &= 1, & \sinh 0 &= 0. \end{aligned}$$