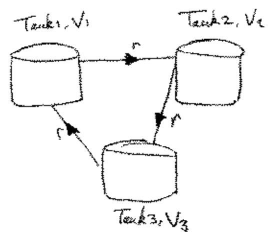
Math 2280-2, Practice Final Exam

April 22, 2008

Total: 165/150 points

Problem 1 (30 pts) Consider the three tank system depicted below.



The tanks have capacities $V_1 = 20$ gal, $V_2 = 50$ gal and $V_3 = 20$ gal and they are assumed to be full at $t = 0$ min. The solution flow rate in the pipes between the tanks is $r = 10$ gal/min. Let $x_i(t)$ be the quantity (pounds) of salt dissolved in the $i-$th tank at time t. Initially ($t = 0$ min) tank 1 contains 18 pounds of salt while the two other tanks contain freshwater.

(a) Derive the first order system satisfied by the system of tanks. Put the system in the form $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))^T$.

Accounting for the rates at which salt enters/leaves the tanks:

$$\begin{cases} \dfrac{dx_1}{dt} = -\dfrac{r}{V_1} x_1 + \dfrac{r}{V_3} x_3 \\ \dfrac{dx_2}{dt} = \dfrac{r}{V_1} x_1 - \dfrac{r}{V_2} x_2 \\ \dfrac{dx_3}{dt} = \dfrac{r}{V_2} x_2 - \dfrac{r}{V_3} x_3 \end{cases}$$

Thus $\underline{x}' = A \underline{x}$ where

$$A = \begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/2 & -1/5 & 0 \\ 0 & 1/5 & -1/2 \end{bmatrix}.$$

note: the calculations in this problem are very long.
the final will not have such a long problem!

(b) Find the eigenvalues and eigenvectors of A.
(**hint:** the eigenvalues are 0 and $-3/5 \pm i3/10$)

$$p(\lambda) = \det(A-\lambda I) = \left(-\tfrac{1}{2}-\lambda\right)\begin{vmatrix} -\frac{1}{5}-\lambda & 0 \\ 1/5 & -1/2-\lambda \end{vmatrix} - \frac{1}{2}\begin{vmatrix} 0 & 1/2 \\ 1/5 & -1/2-\lambda \end{vmatrix} = -\left(\lambda+\tfrac{1}{2}\right)^2\left(\lambda+\tfrac{1}{5}\right)+\tfrac{1}{20}$$

$$= -\left(\lambda^3+\tfrac{6}{5}\lambda^2+\tfrac{9}{20}\lambda\right) = -\lambda\left(\lambda^2+\tfrac{6}{5}\lambda+\tfrac{9}{20}\right) \Rightarrow \text{eigenvalues are } \lambda=0,\ \lambda=-\tfrac{3}{5}\pm\tfrac{3}{10}i$$

For $\lambda_1 = 0$: Find $\underline{v}_1$ s.t. $A\underline{v}_1 = \underline{0}$:

$$\begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{5} & 0 \\ 0 & 1/5 & -1/2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow a = c$

$\frac{a}{2} = \frac{b}{5}$ & $\frac{b}{5} = \frac{c}{2}$

take $\underline{v}_1 = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$

For λ_2 and λ_3 since we have a conjugate eigenvalue pair we can work with $\lambda_2 = -\frac{3}{5} - \frac{3}{10}i$ first:

$$\begin{bmatrix} \frac{1}{10}+\frac{3}{10}i & 0 & 1/2 \\ \frac{1}{2} & \frac{2}{5}+\frac{3}{10}i & 0 \\ 0 & 1/5 & \frac{1}{10}+\frac{3}{10}i \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \boxed{\underline{v}_2 = \begin{bmatrix} 5 \\ -4+3i \\ -1-3i \end{bmatrix}}$$

Then for $\lambda_3 = \bar{\lambda}_2$, $\boxed{\underline{v}_3 = \overline{\underline{v}_2} = \begin{bmatrix} 5 \\ -4-3i \\ -1+3i \end{bmatrix}}$

(c) Find the quantity of salt in each tank as a function of time, with the given initial condition (**hint:** it is quicker **not** to compute the matrix exponential in this case)

We must have: $\underline{x}(t) = \Phi(t)\,\underline{c}$, where $\Phi(t)$ = a fundamental matrix sol.
in order to save us from working w/ cplx qties we use:

$$\Phi(t) = \left[e^{\lambda_1 t}\underline{v}_1,\ \mathrm{Re}(e^{\lambda_2 t}\underline{v}_2),\ \mathrm{Im}(e^{\lambda_2 t}\underline{v}_2)\right] = \begin{bmatrix} 2 & (5\cos\frac{3t}{10})e^{-\frac{3t}{5}} & (-5\sin\frac{3t}{10})e^{-\frac{3t}{5}} \\ 5 & (-4\cos\frac{3t}{10}+3\sin\frac{3t}{10})e^{-\frac{3t}{5}} & (3\cos\frac{3t}{10}+4\sin\frac{3t}{10})e^{-\frac{3t}{5}} \\ 2 & (-\cos\frac{3t}{10}-3\sin\frac{3t}{10})e^{-\frac{3t}{5}} & (-3\cos\frac{3t}{10}+\sin\frac{3t}{10})e^{-\frac{3t}{5}} \end{bmatrix}$$

$$\Phi(0) = \begin{bmatrix} 2 & 5 & 0 \\ 5 & -4 & 3 \\ 2 & -1 & -3 \end{bmatrix} \Rightarrow \underline{c} = \Phi(0)^{-1}\underline{x}(0) = \begin{bmatrix} 2 \\ 14/5 \\ 2/5 \end{bmatrix}$$

$$\boxed{\Rightarrow \underline{x}(t) = \begin{bmatrix} 4 \\ 10 \\ 4 \end{bmatrix} + e^{-\frac{3t}{5}}\begin{bmatrix} 14\cos\frac{3t}{10} - 2\sin\frac{3t}{10} \\ -10\cos\frac{3t}{10} + 10\sin\frac{3t}{10} \\ -4\cos\frac{3t}{10} - 8\sin\frac{3t}{10} \end{bmatrix}}$$

(d) What is the limiting quantity of salt in the tanks?

We have $\lim_{t\to\infty} x(t) = \begin{bmatrix} 4 \\ 10 \\ 4 \end{bmatrix}$.

Problem 2 (25 pts) Solve the following initial value problem using the Laplace transform (**note**: in the real final you will be provided an extract of the Table of Laplace Transforms in the front cover of your book)
(**hint:** $x(t) = (6e^{2t} - 5 - e^{-3t})/15$)

$$\begin{cases} x^{(3)} + x'' - 6x' = 0, \\ x(0) = 0,\ x'(0) = x''(0) = 1. \end{cases}$$

$\xrightarrow{\mathcal{L}}$ $s^3X(s) + s^2X(s) - 6sX(s) = (x''(0) + sx'(0) + x(0)) + (x'(0) + sx(0)) - 6x(0)$

$= 2 + s$

$\Rightarrow X(s) = \dfrac{2+s}{s^3+s^2-6s} = \dfrac{2+s}{s(s-2)(s+3)} = \dfrac{A}{s} + \dfrac{B}{s-2} + \dfrac{C}{s+3}$

$= -\dfrac{1}{3s} + \dfrac{2}{5(s-2)} - \dfrac{1}{15(s+3)}$

$\xrightarrow{\mathcal{L}^{-1}}$ $\boxed{x(t) = -\frac{1}{3} + \frac{2}{5}e^{2t} - \frac{1}{15}e^{-3t}.}$

Problem 3 (25 pts) Consider the logistic population modeled by the differential equation

$$x' = 3x - x^2$$

(a) What are the critical points of the solution?

$3x - x^2 = 0 \Rightarrow x(3-x) = 0$

critical pts are $\boxed{x = 0 \text{ and } x = 3.}$

typo ⚠

(b) Find the population $x(t)$ knowing that $x(0) = x_0 > 0$

The DE is separable: $\displaystyle\int \frac{dx}{x(3-x)} = \int dt = t \Rightarrow \ln \frac{|x|}{|3-x|} = 3t$

$\displaystyle \frac{1}{3}\int \left[\frac{1}{x} + \frac{1}{3-x}\right] dx \qquad \Rightarrow \left|\frac{x}{3-x}\right| = Ce^{3t}$

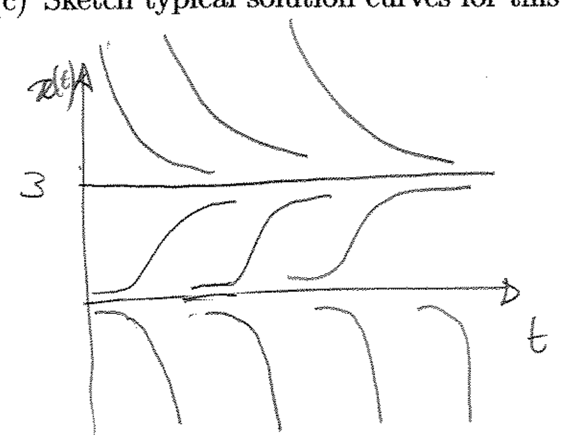
where $C = \left|\frac{x_0}{3-x_0}\right|$

We have two cases:

$0 < x_0 < 3$: $\quad \frac{x_0}{3-x_0} > 0$ and so is $\frac{x}{3-x} \Rightarrow \boxed{x(t) = \frac{3C}{e^{-3t} + C}}$

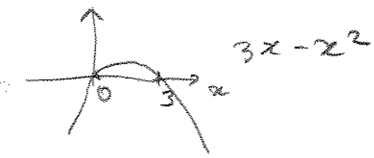
$x_0 > 3$: $\quad \frac{x_0}{3-x_0} < 0$ and so is $\frac{x}{3-x} \Rightarrow \boxed{x(t) = \frac{-3C}{e^{-3t} - C}}$

(c) Sketch typical solution curves for this population.

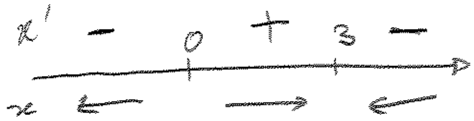


(d) What is the limiting population if $x_0 > 0$?

We have: $\lim_{t \to \infty} x(t) = 3$ regardless of $x_0 > 0$.



(e) Sketch the phase diagram for the population.
Clearly specify the stability of each critical point.



$\underline{x = 0}$: unstable equilibrium

$\underline{x = 3}$: stable equilibrium.

Problem 4 (30 pts) Consider the two population system

$$\begin{cases} x' = 3x - x^2 - \dfrac{xy}{2} \\ y' = 4y - 2xy \end{cases}$$

(a) How does each population behave in the absence of the other one (logistic, exponential growth or decrease)?

x - pop: logistic population (as in previous problem)

y - pop : natural growth.

(b) Describe the nature of the interaction between the x−population and the y−population (competition, cooperation, predation)

The terms modeling interaction indicate that each population's growth is hampered by high levels of the other pop.

$\Rightarrow$ competitive species

(c) Find the three critical points of the population (**hint:** one of them is $(2, 2)^T$, and you essentially found the other two in the previous problem).

We need to solve:

$$\begin{cases} 3x - x^2 - \dfrac{xy}{2} = 0 \\ 4y - 2xy = 0 \end{cases} \iff \begin{cases} x\left(3 - x - \dfrac{y}{2}\right) = 0 \\ \text{and} \\ y(4 - 2x) = 0 \end{cases}$$

$\Rightarrow$ $x = 0, y = 0$

$y = 0, x = 3$

$x = 2, y = 2$

(d) Classify the type and stability of each critical point.

The technique is to look at linearization around critical pt.

$$J(x,y) = \begin{bmatrix} 3-2x-y/2 & -x/2 \\ -2y & 4-2x \end{bmatrix}$$

At $(0,0)$:

$$J(0,0) = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \rightsquigarrow \text{unstable node.} \quad (\lambda = 3, 4)$$

At $(3,0)$:

$$J(3,0) = \begin{bmatrix} -3 & -3/2 \\ 0 & -2 \end{bmatrix} \rightsquigarrow \text{stable node} \quad (\lambda = -3, -2)$$

At $(2,2)$:

$$J(2,2) = \begin{bmatrix} -2 & -1 \\ -4 & 0 \end{bmatrix} \rightsquigarrow \text{saddle pt}$$

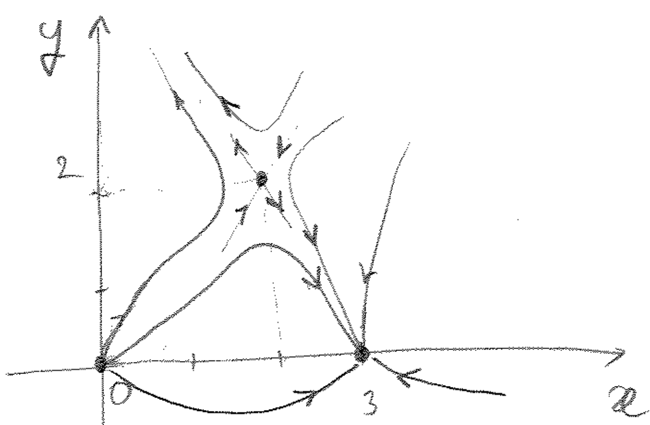
eigenvectors:

$\lambda_1 = -1+\sqrt{5}, \ \underline{v}_1 = (1-\sqrt{5}, 4)^T$

$\lambda_2 = -1-\sqrt{5}, \ \underline{v}_2 = (1+\sqrt{5}, 4)^T$

$$p(\lambda) = (-2-\lambda)(-\lambda) - 2 = \lambda^2 + 2\lambda - 4 \Rightarrow \lambda = -1 \pm \sqrt{5}$$

(e) Sketch the phase portrait of the population system. (**note:** if the point is a saddle point, carefully plot the axis of the hyperbolic trajectories, if the point is a spiral, find the orientation by e.g. computing the tangent field at a couple of points)

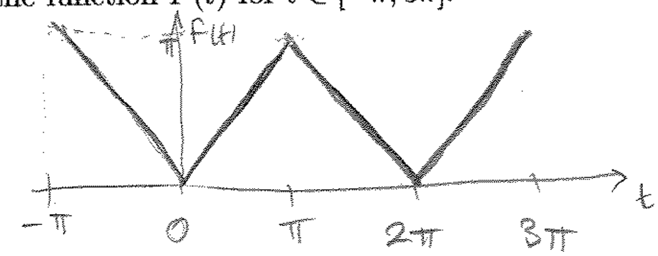


Problem 5 (30 pts) Consider a spring mass system governed by the DE

$$x''(t) + 25x(t) = F(t),$$

where $F(t)$ is a 2π-periodic *even* external force with $F(t) = t$ for $0 < t < \pi$.

(a) Sketch the function $F(t)$ for $t \in [-\pi, 3\pi]$.



(b) Show that the Fourier series of $F(t)$ is

$$F(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n\text{odd}} \frac{\cos nt}{n^2}.$$

We use the even extension formula:

For $n \geq 1$: $\boxed{a_n = \frac{2}{\pi}\int_0^\pi t \cos nt \, dt} \overset{\text{IBP}}{=} \frac{2}{\pi}\frac{1}{n} t \sin nt \Big|_0^\pi - \frac{2}{\pi}\frac{1}{n}\int_0^\pi \sin nt \, dt$

$$= \frac{2}{\pi n^2} \cos nt \Big|_0^\pi = \frac{1}{n^2}\frac{2}{\pi}(\cos n\pi - 1) = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{-4}{\pi n^2} & \text{if } n \text{ odd} \end{cases}$$

For $n = 0$: $\boxed{a_0 = \frac{2}{\pi}\int_0^\pi t \, dt = \frac{2}{\pi}\frac{\pi^2}{2} = \frac{\pi}{2}}$

$$\boxed{\text{Thus } F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n\text{ odd}} \frac{\cos nt}{n^2}}$$

(c) Use the Laplace transform method to find the steady periodic (particular) solution to

$$x'' + 25x = \cos 5t.$$

What can you say about the magnitude of the solution? How is this phenomenon called?

For simplicity we take $x(0) = x'(0) = 0$. Thus:

$$\xrightarrow{\mathcal{L}} \quad s^2 X(s) + 25X(s) = \frac{s}{s^2+25} \Rightarrow X(s) = \frac{s}{(s^2+25)^2}$$

$$\xrightarrow{\mathcal{L}^{-1}} \quad \boxed{x(t) = \frac{t}{10}\sin 5t}$$

we have $|x(t)| \to \infty$ as $t \to \infty$

this is resonance.

(d) Use Fourier series and your previous answer to find ~~the~~ a steady periodic solution to

$$x''(t) + 25x(t) = F(t).$$

We look for a solution of the form: $x_p(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$.

thus the DE becomes:

$$x_p'' + 25x_p = \frac{25}{2}a_0 + \sum_{n=1}^{\infty} a_n(25-n^2)\cos nt$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n^2} \cos nt$$

We then get: $a_0 = \frac{\pi}{25}$, and for $n \neq 5$: $a_n = \begin{cases} \frac{-4}{\pi} \frac{1}{(25-n^2)n^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

The $n=5$ term needs to be treated separately (see (c))

Thus: $x_p(t) = \frac{\pi}{50} - \frac{4}{\pi}\sum_{\substack{n=1\\ n \text{ odd}\\ n\neq 5}}^{\infty} \frac{\cos nt}{(25-n^2)n^2} - \underbrace{\frac{(4/\pi)}{25}}_{n=5 \text{ term in Fourier series of } F(t)} \underbrace{\frac{t}{10}\sin 5t}_{\text{resonance sol. from (c)}}$

Problem 6 (25 pts) Solve the boundary value problem (1D heat equation):

$$\begin{cases} \dfrac{\partial u}{\partial t} = 2\dfrac{\partial^2 u}{\partial x^2}, & \text{for } 0 < x < 1 \text{ and } t > 0, \\ u(0,t) = u(1,t) = 0, & \text{for } t > 0, \\ u(x,0) = 5\sin \pi x - \dfrac{1}{5}\sin 3\pi x, & \text{for } 0 < x < 1. \end{cases}$$

We let $u(x,t) = X(x)T(t) \Rightarrow XT' = 2X''T \Rightarrow \frac{T'}{2T} = \frac{X''}{X} = -\lambda = \text{const.}$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(1) = 0 \end{cases} \Rightarrow \begin{array}{l} X_n(x) = \sin n\pi x \\ \lambda_n = (n\pi)^2 \end{array}$$

$$T_n' + 2\lambda_n T_n = 0 \Rightarrow T_n(t) = e^{-2(n\pi)^2 t}$$

Thus $u(x,t) = \sum_{n=1}^{\infty} b_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} b_n \sin n\pi x \, e^{-2(n\pi)^2 t}$

Using init. cond:

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin n\pi x = 5\sin \pi x - \frac{1}{5}\sin 3\pi x$$

$\Rightarrow b_1 = 5, \; b_3 = -\frac{1}{5}$, and $b_n = 0$ for $n \notin \{1,3\}$.

$$\Rightarrow \boxed{u(x,t) = 5\sin \pi x \, e^{-2\pi^2 t} - \frac{1}{5}\sin 3\pi x \, e^{-2(3\pi)^2 t}}$$