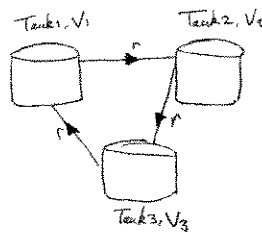


Math 2280-2, Practice Final Exam

April 22, 2008

Total: 165/150 points

Problem 1 (30 pts) Consider the three tank system depicted below.



The tanks have capacities $V_1 = 20$ gal, $V_2 = 50$ gal and $V_3 = 20$ gal and they are assumed to be full at $t = 0$ min. The solution flow rate in the pipes between the tanks is $r = 10$ gal/min. Let $x_i(t)$ be the quantity (pounds) of salt dissolved in the i -th tank at time t . Initially ($t = 0$ min) tank 1 contains 18 pounds of salt while the two other tanks contain freshwater.

- (a) Derive the first order system satisfied by the system of tanks. Put the system in the form $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))^T$.

Accounting for the rates at which salt enters/leaves the tanks:

$$\begin{cases} \frac{dx_1}{dt} = -\frac{r}{V_1} x_1 + \frac{r}{V_3} x_3 \\ \frac{dx_2}{dt} = \frac{r}{V_1} x_1 - \frac{r}{V_2} x_2 \\ \frac{dx_3}{dt} = \frac{r}{V_2} x_2 - \frac{r}{V_3} x_3 \end{cases} \quad \text{Thus } \underline{x}' = A \underline{x} \text{ where}$$

$$A = \begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/2 & -1/5 & 0 \\ 0 & 1/5 & -1/2 \end{bmatrix}.$$

note: the calculations in this problem are very long.
the final will not have such a long problem!

(b) Find the eigenvalues and eigenvectors of A.

(hint: the eigenvalues are 0 and $-3/5 \pm i3/10$)

$$\begin{aligned} \chi(\lambda) = \det(A - \lambda I) &= \begin{vmatrix} -\frac{1}{2} - \lambda & 0 & 0 \\ \frac{1}{\sqrt{5}} & -\frac{1}{2} - \lambda & 0 \\ 0 & \frac{1}{\sqrt{5}} & -\frac{1}{2} - \lambda \end{vmatrix} = -(\lambda + \frac{1}{2})^2 (\lambda + \frac{1}{5}) + \frac{1}{20} \\ &= -(\lambda^3 + \frac{6}{5}\lambda^2 + \frac{9}{20}\lambda) = -\lambda(\lambda^2 + \frac{6}{5}\lambda + \frac{9}{20}) \Rightarrow \text{eigenvalues are } \lambda = 0, \lambda = -\frac{3}{5} \pm \frac{3}{10}i \end{aligned}$$

For $\lambda_1 = 0$: Find \underline{v}_1 s.t. $A\underline{v}_1 = 0$:

$$\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow a = c \\ \frac{a}{2} = \frac{b}{5} \quad \& \quad \frac{b}{5} = \frac{c}{2}$$

$$\text{take } \underline{v}_1 = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$$

For λ_2 and λ_3 since we have a conjugate eigenvalue pair we can work with $\lambda_2 = -\frac{3}{5} - \frac{3}{10}i$ first:

$$\begin{bmatrix} \frac{1}{10} + \frac{3}{10}i & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{2}{5} + \frac{3}{10}i & 0 \\ 0 & \frac{1}{5} & \frac{1}{10} + \frac{3}{10}i \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underline{v}_2 = \begin{bmatrix} 5 \\ -4 + 3i \\ -1 - 3i \end{bmatrix}$$

$$\text{Then for } \lambda_3 = \bar{\lambda}_2, \underline{v}_3 = \bar{\underline{v}}_2 = \begin{bmatrix} 5 \\ -4 - 3i \\ -1 + 3i \end{bmatrix}$$

(c) Find the quantity of salt in each tank as a function of time, with the given initial condition (hint: it is quicker **not** to compute the matrix exponential in this case)

We must have: $\underline{x}(t) = \underline{\Phi}(t)\underline{c}$, where $\underline{\Phi}(t)$ is a fundamental matrix sol.
in order to save us from working w/ cplx qties we use:

$$\underline{\Phi}(t) = \left[e^{\lambda_1 t} \underline{v}_1, \operatorname{Re}(e^{\lambda_2 t} \underline{v}_2), \operatorname{Im}(e^{\lambda_2 t} \underline{v}_2) \right] = \begin{bmatrix} 2 & 5 \cos \frac{3t}{10} & 5 \sin \frac{3t}{10} \\ 5 & (-4 \cos \frac{3t}{10} + 3 \sin \frac{3t}{10}) e^{-\frac{3t}{5}} & (-5 \sin \frac{3t}{10}) e^{-\frac{3t}{5}} \\ 2 & (-\cos \frac{3t}{10} - 3 \sin \frac{3t}{10}) e^{-\frac{3t}{5}} & (3 \cos \frac{3t}{10} + 4 \sin \frac{3t}{10}) e^{-\frac{3t}{5}} \end{bmatrix}$$

$$\underline{\Phi}(0) = \begin{bmatrix} 2 & 5 & 0 \\ 5 & -4 & 3 \\ 2 & -1 & -3 \end{bmatrix} \Rightarrow \underline{c} = \underline{\Phi}(0)^{-1} \underline{x}(0) = \begin{bmatrix} 2 \\ 14/5 \\ 2/5 \end{bmatrix}$$

$$\Rightarrow \underline{x}(t) = \begin{bmatrix} 4 \\ 10 \\ 4 \end{bmatrix} + e^{-\frac{3t}{5}} \begin{bmatrix} 14 \cos \frac{3t}{10} & -2 \sin \frac{3t}{10} \\ -10 \cos \frac{3t}{10} & +10 \sin \frac{3t}{10} \\ -4 \cos \frac{3t}{10} & -8 \sin \frac{3t}{10} \end{bmatrix}$$

(d) What is the limiting quantity of salt in the tanks?

$$\text{We have } \lim_{t \rightarrow \infty} \underline{x}(t) = \begin{bmatrix} 4 \\ 10 \\ 4 \end{bmatrix}.$$

Problem 2 (25 pts) Solve the following initial value problem using the Laplace transform
(note: in the real final you will be provided an extract of the Table of Laplace Transforms in the front cover of your book)

(hint: $x(t) = (6e^{2t} - 5 - e^{-3t})/15$)

$$\begin{cases} x^{(3)} + x'' - 6x' = 0, \\ x(0) = 0, \quad x'(0) = x''(0) = 1. \end{cases}$$

$$\begin{aligned} \xrightarrow{\mathcal{L}} \quad s^3 X(s) + s^2 X(s) - 6s X(s) &= (x''(0) + s x'(0) + x(0)) + (x'(0) + s x(0)) - 6x(0) \\ &= 2 + s \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad X(s) &= \frac{2+s}{s^3 + s^2 - 6s} = \frac{2+s}{s(s-2)(s+3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+3} \\ &= -\frac{1}{3s} + \frac{2}{5(s-2)} - \frac{1}{15(s+3)} \end{aligned}$$

$$\xrightarrow{\mathcal{L}^{-1}} \quad \boxed{x(t) = -\frac{1}{3} + \frac{2}{5} e^{2t} - \frac{1}{15} e^{-3t}}$$

Problem 3 (25 pts) Consider the logistic population modeled by the differential equation

$$x' = 3x - x^2$$

(a) What are the critical points of the solution?

$$3x - x^2 = 0 \Rightarrow x(3-x) = 0$$

critical pts are $x=0$ and $x=3$.

typo \triangle

(b) Find the population $x(t)$ knowing that $x(0) = x_0 > 0$

The DE is separable: $\int \frac{dx}{x(3-x)} = \int dt = t \Rightarrow \ln \frac{|x|}{|3-x|} = 3t$

$$\frac{1}{3} \int \left[\frac{1}{x} + \frac{1}{3-x} \right] dx \Rightarrow \left| \frac{x}{3-x} \right| = C e^{3t}$$

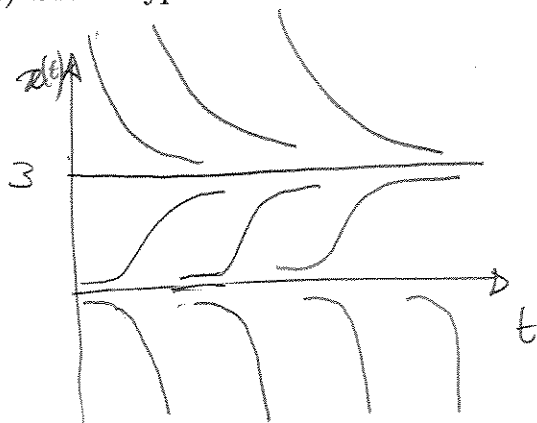
where $C = \left| \frac{x_0}{3-x_0} \right|$

We have two cases:

$0 < x_0 < 3$: $\frac{x_0}{3-x_0} > 0$ and so is $\frac{x}{3-x} \Rightarrow x(t) = \frac{3C}{e^{-3t} + C}$

$x_0 > 3$: $\frac{x_0}{3-x_0} < 0$ and so is $\frac{x}{3-x} \Rightarrow x(t) = \frac{-3C}{e^{-3t} - C}$

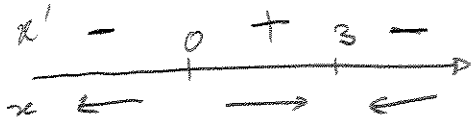
(c) Sketch typical solution curves for this population.



(d) What is the limiting population if $x_0 > 0$?

We have: $\lim_{t \rightarrow \infty} x(t) = 3$ regardless of $x_0 > 0$.

(e) Sketch the phase diagram for the population.
Clearly specify the stability of each critical point.



$x=0$: unstable equilibrium
 $x=3$: stable equilibrium.

Problem 4 (30 pts) Consider the two population system

$$\begin{cases} x' = 3x - x^2 - \frac{xy}{2} \\ y' = 4y - 2xy \end{cases}$$

(a) How does each population behave in the absence of the other one (logistic, exponential growth or decrease)?

x -pop: logistic population (as in previous problem)

y -pop: natural growth.

(b) Describe the nature of the interaction between the x -population and the y -population (competition, cooperation, predation)

The terms modeling interaction indicate that each population's growth is hampered by high levels of the other pop.

\Rightarrow competitive species

(c) Find the three critical points of the population (**hint**: one of them is $(2, 2)^T$, and you essentially found the other two in the previous problem).

We need to solve:

$$\begin{cases} 3x - x^2 - \frac{xy}{2} = 0 \\ 4y - 2xy = 0 \end{cases} \quad (\Leftrightarrow) \quad \begin{cases} x(3 - x - \frac{y}{2}) = 0 \\ y(4 - 2x) = 0 \end{cases}$$

$$\Rightarrow x=0, y=0$$

$$y=0, x=3$$

$$x=2, y=2$$

(d) Classify the type and stability of each critical point.

The technique is to look at linearization around critical pt.

$$J(x,y) = \begin{bmatrix} 3-2x-y & -x \\ -2y & 4-2x \end{bmatrix}$$

At (0,0):

$$J(0,0) = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \rightarrow \text{unstable node} \quad (\lambda = 3, 4)$$

At (3,0):

$$J(3,0) = \begin{bmatrix} -3 & -3 \\ 0 & -2 \end{bmatrix} \rightarrow \text{stable node} \quad (\lambda = -3, -2)$$

At (2,2):

$$J(2,2) = \begin{bmatrix} -2 & -1 \\ -4 & 0 \end{bmatrix} \rightarrow \text{saddle pt}$$

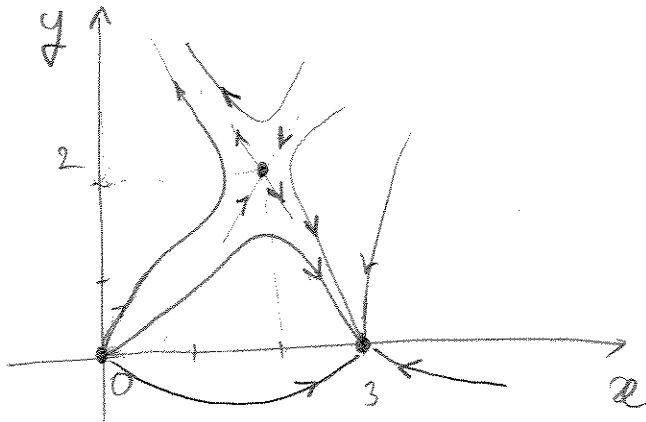
eigenvectors:

$$\lambda_1 = -1 + \sqrt{5}, \quad \underline{v}_1 = (1 - \sqrt{5}, 4)^T$$

$$\lambda_2 = -1 - \sqrt{5}, \quad \underline{v}_2 = (1 + \sqrt{5}, 4)^T$$

$$p(\lambda) = (-2-\lambda)(-\lambda) - 2 = \lambda^2 + 2\lambda - 4 \Rightarrow \lambda = -1 \pm \sqrt{5}$$

(e) Sketch the phase portrait of the population system. (note: if the point is a saddle point, carefully plot the axis of the hyperbolic trajectories, if the point is a spiral, find the orientation by e.g. computing the tangent field at a couple of points)

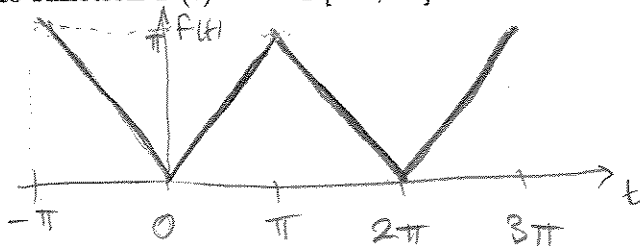


Problem 5 (30 pts) Consider a spring mass system governed by the DE

$$x''(t) + 25x(t) = F(t),$$

where $F(t)$ is a 2π -periodic *even* external force with $F(t) = t$ for $0 < t < \pi$.

(a) Sketch the function $F(t)$ for $t \in [-\pi, 3\pi]$.



(b) Show that the Fourier series of $F(t)$ is

$$F(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos nt}{n^2}.$$

We use the even extension formula:

For $n \geq 1$:
$$a_n = \frac{2}{\pi} \int_0^{\pi} t \cos nt \, dt \stackrel{\text{IBP}}{=} \frac{2}{\pi n} t \sin nt \Big|_0^{\pi} - \frac{2}{\pi n} \int_0^{\pi} \sin nt \, dt$$

$$= \frac{2}{\pi n} \cos nt \Big|_0^{\pi} = \frac{1}{n^2} \frac{2}{\pi} (\cos n\pi - 1) = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{4}{\pi n^2} & \text{if } n \text{ odd} \end{cases}$$

For $n=0$:
$$a_0 = \frac{2}{\pi} \int_0^{\pi} t \, dt = \frac{2}{\pi} \frac{\pi^2}{2} = \frac{\pi}{2}$$

Thus
$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos nt}{n^2}$$

(c) Use the Laplace transform method to find the steady periodic (particular) solution to

$$x'' + 25x = \cos 5t.$$

What can you say about the magnitude of the solution? How is this phenomenon called?

For simplicity we take $x(0) = x'(0) = 0$. Thus:

$$\mathcal{L}\{x'' + 25x\} = \mathcal{L}\{\cos 5t\} \Rightarrow X(s) = \frac{1}{(s^2 + 25)^2}$$

$$\mathcal{L}^{-1}\{X(s)\} \Rightarrow x(t) = \frac{t}{10} \sin 5t$$

we have $|x(t)| \rightarrow \infty$ as $t \rightarrow \infty$
this is resonance.

(d) Use Fourier series and your previous answer to find the steady periodic solution to

$$x''(t) + 25x(t) = F(t).$$

We look for a solution of the form: $x_p(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$.

Thus the DE becomes:

$$\begin{aligned} x_p'' + 25x_p &= \frac{25}{2} a_0 + \sum_{n=1}^{\infty} a_n (25 - n^2) \cos nt \\ &= \frac{\pi}{2} - \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2} \cos nt \end{aligned}$$

We then get: $a_0 = \frac{\pi}{25}$, and for $n \neq 5$: $a_n = \begin{cases} \frac{-4}{\pi} \frac{1}{(25 - n^2)n^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

The $n=5$ term needs to be treated separately (see (c)).

Thus: $x_p(t) = \frac{\pi}{50} - \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd} \\ n \neq 5}}^{\infty} \frac{\cos nt}{(25 - n^2)n^2} - \underbrace{\left(\frac{4}{25}\right)}_{\substack{n=5 \text{ term in} \\ \text{Fourier series of } F(t)}} \underbrace{\frac{t}{10} \sin 5t}_{\substack{\text{resonance sol.} \\ \text{from (c)}}$

Problem 6 (25 pts) Solve the boundary value problem (1D heat equation):

$$\begin{cases} \frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2}, & \text{for } 0 < x < 1 \text{ and } t > 0, \\ u(0, t) = u(1, t) = 0, & \text{for } t > 0, \\ u(x, 0) = 5 \sin \pi x - \frac{1}{5} \sin 3\pi x, & \text{for } 0 < x < 1. \end{cases}$$

We let $u(x, t) = X(x)T(t) \Rightarrow X'T' = 2X''T \Rightarrow \frac{T'}{2T} = \frac{X''}{X} = -\lambda = \text{const.}$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(1) = 0 \end{cases} \Rightarrow \begin{cases} X_n(x) = \sin n\pi x \\ \lambda_n = (n\pi)^2 \end{cases}$$

$$T_n' + 2\lambda_n T_n = 0 \Rightarrow T_n(t) = e^{-2(n\pi)^2 t}$$

Thus $u(x, t) = \sum_{n=1}^{\infty} b_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} b_n \sin n\pi x e^{-2(n\pi)^2 t}$

Using init. cond:

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin n\pi x = 5 \sin \pi x - \frac{1}{5} \sin 3\pi x$$

$$\Rightarrow b_1 = 5, b_3 = -\frac{1}{5}, \text{ and } b_n = 0 \text{ for } n \notin \{1, 3\}.$$

$$\Rightarrow \boxed{u(x, t) = 5 \sin \pi x e^{-2\pi^2 t} - \frac{1}{5} \sin 3\pi x e^{-2(3\pi)^2 t}}$$