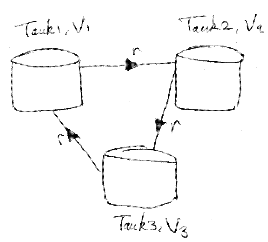
Math 2280-2, Practice Final Exam

April 22, 2008

Total: 165/150 points

Problem 1 (30 pts) Consider the three tank system depicted below.



The tanks have capacities $V_1 = 20$ gal, $V_2 = 50$ gal and $V_3 = 20$ gal and they are assumed to be full at $t = 0$ min. The solution flow rate in the pipes between the tanks is $r = 10$ gal/min. Let $x_i(t)$ be the quantity (pounds) of salt dissolved in the $i-$th tank at time t. Initially ($t = 0$ min) tank 1 contains 18 pounds of salt while the two other tanks contain freshwater.

(a) Derive the first order system satisfied by the system of tanks. Put the system in the form $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))^T$.

(b) Find the eigenvalues and eigenvectors of A.
(**hint:** the eigenvalues are 0 and $-3/5 \pm i3/10$)

(c) Find the quantity of salt in each tank as a function of time, with the given initial condition (**hint:** it is quicker **not** to compute the matrix exponential in this case)

(d) What is the limiting quantity of salt in the tanks?

Problem 2 (25 pts) Solve the following initial value problem using the Laplace transform (**note**: in the real final you will be provided an extract of the Table of Laplace Transforms in the front cover of your book)
(**hint:** $x(t) = (6e^{2t} - 5 - e^{-3t})/15$)

$$\begin{cases} x^{(3)} + x'' - 6x' = 0, \\ x(0) = 0, \ x'(0) = x''(0) = 1. \end{cases}$$

Problem 3 (25 pts) Consider the logistic population modeled by the differential equation

$$x' = 3x - x^2$$

(a) What are the critical points of the solution?

(b) Find the population $x(t)$ knowing that $x(0) = x_0$.

(c) Sketch typical solution curves for this population.

(d) What is the limiting population if $x_0 > 0$?

(e) Sketch the phase diagram for the population.
Clearly specify the stability of each critical point.

Problem 4 (30 pts) Consider the two population system

$$\begin{cases} x' = 3x - x^2 - \dfrac{xy}{2} \\ y' = 4y - 2xy \end{cases}$$

(a) How does each population behave in the absence of the other one (logistic, exponential growth or decrease)?

(b) Describe the nature of the interaction between the $x-$population and the $y-$population (competition, cooperation, predation)

(c) Find the three critical points of the population (**hint:** one of them is $(2,2)^T$, and you essentially found the other two in the previous problem).

(d) Classify the type and stability of each critical point.

(e) Sketch the phase portrait of the population system. (**note:** if the point is a saddle point, carefully plot the axis of the hyperbolic trajectories, if the point is a spiral, find the orientation by e.g. computing the tangent field at a couple of points)

Problem 5 (30 pts) Consider a spring mass system governed by the DE

$$x''(t) + 25x(t) = F(t),$$

where $F(t)$ is a 2π-periodic *even* external force with $F(t) = t$ for $0 < t < \pi$.

(a) Sketch the function $F(t)$ for $t \in [-\pi, 3\pi]$.

(b) Show that the Fourier series of $F(t)$ is

$$F(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n\text{odd}} \frac{\cos nt}{n^2}.$$

(c) Use the Laplace transform method to find the steady periodic (particular) solution to

$$x'' + 25x = \cos 5t.$$

What can you say about the magnitude of the solution? How is this phenomenon called?

(d) Use Fourier series and your previous answer to find the steady periodic solution to

$$x''(t) + 25x(t) = F(t).$$

Problem 6 (25 pts) Solve the boundary value problem (1D heat equation):

$$\begin{cases} \dfrac{\partial u}{\partial t} = 2\dfrac{\partial^2 u}{\partial x^2}, & \text{for } 0 < x < 1 \text{ and } t > 0, \\ u(0,t) = u(1,t) = 0, & \text{for } t > 0, \\ u(x,0) = 5\sin \pi x - \dfrac{1}{5}\sin 3\pi x, & \text{for } 0 < x < 1. \end{cases}$$