

SOLUTIONS

MATH 2280-2, EXAM 2
APRIL 3 2008

Total: 100 points

Problem 1 (30 pts) Consider the first order system of differential equations $x' = Ax$, where

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A . (hint: the eigenvalues are -1 and 5 .)

$$P(\lambda) = \det(A - \lambda I) = (2-\lambda)(2-\lambda) - 9 = \lambda^2 - 4\lambda - 5$$

$$\Rightarrow \lambda = \frac{4 \pm \sqrt{16 + 20}}{2} = \begin{cases} 5 \\ -1 \end{cases}$$

For $\lambda_1 = 5$ eigenvector is:

$$(A - 5I)u = 0 \Leftrightarrow$$

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \underline{v}_1 = (1, 1)^T$$

For $\lambda_2 = -1$ eigenvector is:

$$(A + I)u = 0 \Leftrightarrow$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \underline{v}_2 = (1, -1)^T$$

(b) Find a fundamental matrix solution $\Phi(t)$ to the system $x' = Ax$.

$$\Phi(t) = \begin{bmatrix} \underline{x}_1(t) & \underline{x}_2(t) \\ | & | \end{bmatrix}$$

where $\underline{x}_1(t) = e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
 $\underline{x}_2(t) = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$= \begin{bmatrix} e^{5t} & e^{-t} \\ e^{5t} & -e^{-t} \end{bmatrix}.$$

(c) Find the matrix exponential e^{At} . Using your answer, find the solution to $\mathbf{x}' = A\mathbf{x}$, with $\mathbf{x}(0) = (1, 0)^T$.

$$\begin{aligned}
 e^{At} &= \Phi(t) \Phi(0)^{-1} = \begin{bmatrix} e^{st} & e^{-t} \\ e^{st} & -e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \\
 &= -\frac{1}{2} \begin{bmatrix} e^{st} & e^{-t} \\ e^{st} & -e^{-t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\
 &= \frac{1}{2} \begin{bmatrix} e^{st} + e^{-t} & e^{st} - e^{-t} \\ e^{st} - e^{-t} & e^{st} + e^{-t} \end{bmatrix}
 \end{aligned}$$

(d) Find a particular solution $\mathbf{x}_p(t)$ to $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, where $\mathbf{f} = (4, 6)^T$. (hint: Use $\mathbf{x}_p(t) = \mathbf{u} = \text{constant vector}$.)

$$\begin{aligned}
 \mathbf{0} = \mathbf{x}_p' &= A\mathbf{x}_p + \mathbf{f} \Rightarrow \mathbf{x}_p = -A^{-1}\mathbf{f} \\
 &= -\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 6 \end{bmatrix} \\
 &= -\frac{1}{-5} \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} \\
 &= \frac{1}{5} \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}
 \end{aligned}$$

(e) Solve the initial value problem $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, with $\mathbf{x}(0) = (1, -1)^T$, and the same \mathbf{f} as above.

$$\mathbf{x}(t) = \mathbf{x}_p(t) + \mathbf{x}_h(t) = \begin{bmatrix} -2 \\ 0 \end{bmatrix} + \Phi(t) \begin{bmatrix} a \\ b \end{bmatrix}$$

↑ could also use matrix exp.

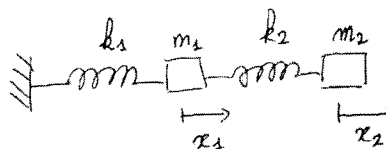
at $t=0$:

$$\begin{aligned}
 \begin{pmatrix} 1 \\ -1 \end{pmatrix} &= \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \Phi(0) \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \Phi(0)^{-1} \begin{pmatrix} 3 \\ -1 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ -1 \end{pmatrix} \\
 &= \frac{-1}{2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} \\
 &= \begin{pmatrix} 1 \\ 2 \end{pmatrix}
 \end{aligned}$$

thus:

$$\mathbf{x}(t) = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + e^{st} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Problem 2 (30 pts) Consider the two masses two springs system depicted below.



- (a) Derive the second order system of DEs satisfied by $x_1(t)$ and $x_2(t)$ (displacements around the equilibrium position), assuming there are no external forces. Put the system in the form $M\mathbf{x}'' = K\mathbf{x}$, specifying what the matrices M and K are.

$$\begin{cases} m_1 x_1'' = -k_1 x_1 + k_2 (x_2 - x_1) \\ m_2 x_2'' = +k_2 (x_1 - x_2) \end{cases}$$

$$\Leftrightarrow M\mathbf{x}'' = K\mathbf{x} \text{ where}$$

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad K = \begin{bmatrix} -(k_1 + k_2) & k_2 \\ k_2 & -k_2 \end{bmatrix}.$$

- (b) With $m_1 = 1$, $m_2 = 1/2$, $k_1 = 2$ and $k_2 = 1$ show that the system is equivalent to $\mathbf{x}'' = A\mathbf{x}$, with

$$A = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}.$$

$$A = M^{-1}K = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix}.$$

- (c) Find the general solution to the system $\mathbf{x}'' = A\mathbf{x}$.

$$P(\lambda) = (-3 - \lambda)(-2 - \lambda) - 2 = \lambda^2 + 5\lambda + 4. \quad \lambda = \frac{-5 \pm \sqrt{25 - 16}}{2} = \frac{-5 \pm 3}{2} = \begin{cases} -1 \\ -4 \end{cases}$$

eigenvectors:

$$\begin{aligned} \lambda_1 = -1 & \\ (A + I)\mathbf{u} = \mathbf{0} \Leftrightarrow \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \left| \begin{aligned} \lambda_2 = -4 \\ (A + 4I)\mathbf{u} = \mathbf{0} \Leftrightarrow \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \Rightarrow \underline{v}_2 = (1, -1)^T. \end{aligned} \right. \\ \Rightarrow \underline{v}_1 = (1, 2)^T. \end{aligned}$$

$$\Rightarrow \mathbf{x}(t) = (a \cos \omega_1 t + b \sin \omega_1 t) \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (c \cos \omega_2 t + d \sin \omega_2 t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ where } \omega_1 = \sqrt{-\lambda_1} = 1, \omega_2 = \sqrt{-\lambda_2} = 2.$$

- (d) What are the natural frequencies of the system $\mathbf{x}'' = A\mathbf{x}$? Describe their corresponding modes of oscillation.

$\omega_1 = 1$ angular freq: masses move in phase, with second mass oscillating with twice the amplitude as first one.

$\omega_2 = 2$ masses move in opposite phases with same amplitude.

- (e) Find a particular solution to the system $\mathbf{x}'' = A\mathbf{x} + \mathbf{f}$, where the external force is $\mathbf{f}(t) = (\cos \omega t, 0)^T$, and ω is not a resonance frequency of $\mathbf{x}'' = A\mathbf{x}$.

$\mathbf{x}_p(t) = \underline{\mathbf{a}} \cos \omega t$. Plugging in the DE:

$$-\omega^2 \cos \omega t \underline{\mathbf{a}} = \cos \omega t A \underline{\mathbf{a}} + \cos \omega t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow (-\omega^2 I + A) \underline{\mathbf{a}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \underline{\mathbf{a}} = - \begin{bmatrix} -3 + \omega^2 & 1 \\ 2 & -2 + \omega^2 \end{bmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{p(-\omega^2)} \begin{bmatrix} -2 + \omega^2 & -1 \\ -2 & -3 + \omega^2 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where $p(\lambda) = \lambda^2 + 5\lambda + 4$

= char poly

$\neq 0$ when $\lambda \neq -\omega_i^2$

(not resonant freq).

$$= \frac{1}{p(-\omega^2)} \begin{bmatrix} 2 - \omega^2 \\ 2 \end{bmatrix}$$

$$\Rightarrow \boxed{\underline{\mathbf{x}}_p(t) = \frac{\cos \omega t}{p(-\omega^2)} \begin{bmatrix} 2 - \omega^2 \\ 2 \end{bmatrix}}$$

Problem 3 (10 pts) Consider the matrix

$$N = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Show N is nilpotent.

$$N^2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N^3 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

\Rightarrow Not Nilpotent.

(b) Use your answer to compute the matrix exponential e^{Nt} .

$$e^{Nt} = I + tN + \frac{t^2}{2} N^2 = \begin{bmatrix} 1 & t & 2t + t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 4 (30 pts) Consider the two populations system,

$$\begin{cases} \frac{dx}{dt} = 3x - x^2 - \frac{1}{4}xy \\ \frac{dy}{dt} = xy - 2y. \end{cases}$$

- (a) How does each population behave in the absence of the other one (logistic, exponential growth or decrease)? Describe the nature of the interaction between the x -population and the y -population (competition, cooperation, predation).

x -pop. is logistic when isolated from y -pop.

y -pop. exponentially decreases by itself.

moreover: y -pop benefits from increase in x -pop
 x -pop ————— decrease in y -pop

\Rightarrow predator/prey interaction where

x -pop = prey

y -pop = predator.

- (b) Find the three critical points of the system (hint: one critical point is $(2, 4)^T$.)

$$\begin{cases} 3x - x^2 - \frac{xy}{4} = 0 \\ xy - 2y = 0 \end{cases} \Leftrightarrow \begin{cases} x(3 - x - y/4) = 0 \\ \text{and} \\ y(x - 2) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0, y = 0 \\ x = 2, y = 4 \\ y = 0, x = 3 \end{cases}$$

Critical pts are $(0, 0)$, $(2, 4)$ and $(3, 0)$

(c) Find the linearized system around the critical point $(2, 4)^T$.

$$J(x, y) = \begin{bmatrix} 3 - 2x - y/4 & -x/4 \\ y & x - 2 \end{bmatrix}$$

$$J(2, 4) = \begin{bmatrix} -2 & -1/2 \\ 4 & 0 \end{bmatrix}$$

\Rightarrow Linearized system around $(2, 4)^T$ is:

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{bmatrix} -2 & -1/2 \\ 4 & 0 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{where } \begin{cases} x = 2 + u \\ y = 4 + v \end{cases}$$

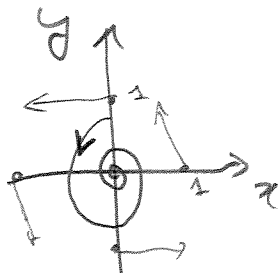
(d) Sketch the phase portrait for the linearized problem. (if the point is a saddle point, carefully plot the axis of the hyperbolic trajectories, if the point is a spiral, find the orientation by e.g. computing the tangent field at a couple of points). What type of critical point is this? Is it stable or unstable?

$$p(\lambda) = (-2 - \lambda)(-\lambda) + 2 = \lambda^2 + 2\lambda + 2$$

char poly of $J(2, 4)$

$$\lambda = \frac{-2 \pm \sqrt{4 - 8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

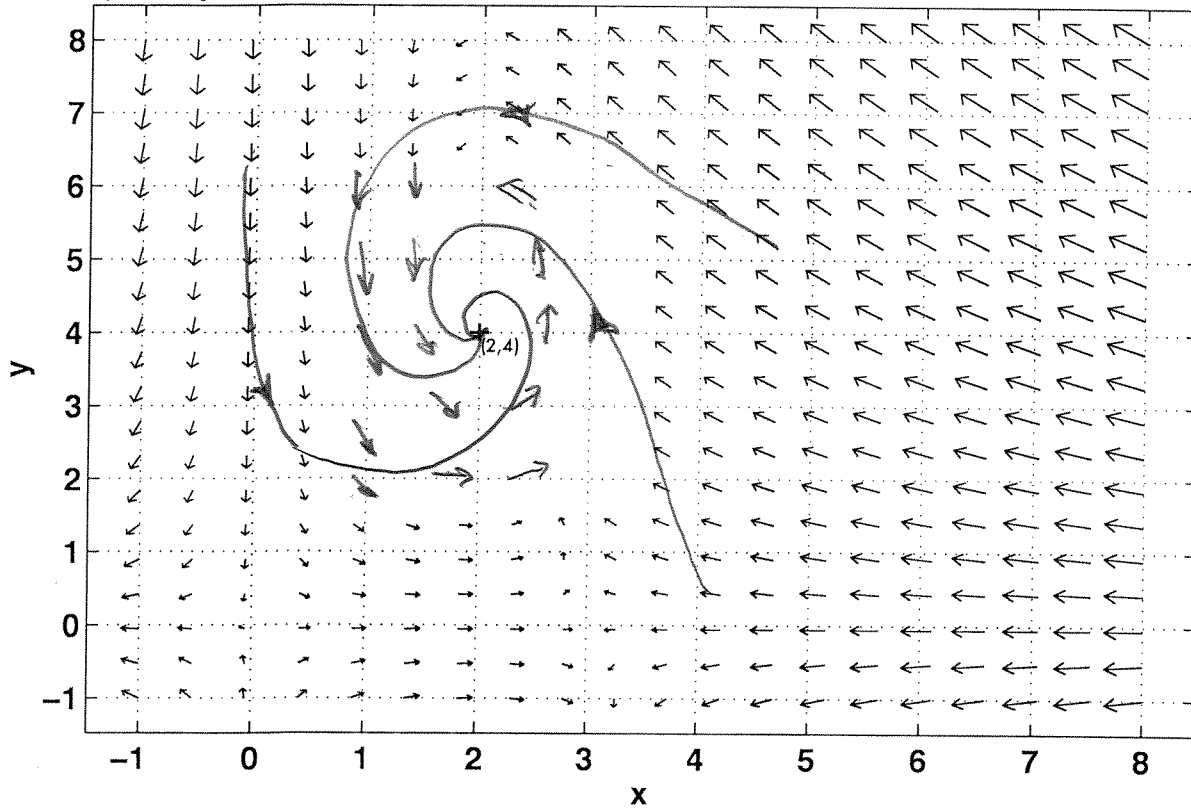
We have a complex conjugate pair of eigenvalues w/ negative real part $\Rightarrow (0, 0)$ is a stable spiral for lin. prob.



- (e) Complete the phase portrait below around the critical point $(2, 4)^T$. When $x(0) > 0$ and $y(0) > 0$, what do you expect about the x and y populations after a large time? Sketch a few trajectories representative of your predictions about the populations.

$$x' = 3x - x^2 - xy/4$$

$$y' = xy - 2y$$



If the starting population is s.t. $x(0) > 0$ and $y(0) > 0$ then as $t \rightarrow \infty$ we expect the populations to converge to the stable critical pt $x = 2, y = 4$ as illustrated above.