$\boxed{\text{SOLUTIONS}}$

MATH 2280-2, EXAM 2
APRIL 3 2008

Total: 100 points
Problem 1 (30 pts) Consider the first order system of differential equations $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A. (**hint:** the eigenvalues are -1 and 5.)

$$p(\lambda) = \det(A - \lambda I) = (2-\lambda)(2-\lambda) - 9 = \lambda^2 - 4\lambda - 5$$

$$\Rightarrow \lambda = \frac{4 \pm \sqrt{16+20}}{2} = \begin{cases} 5 \\ -1 \end{cases}$$

For $\lambda_1 = 5$ eigenvector is:

$$(A - 5I)\underline{u} = \underline{0} \Leftrightarrow$$

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \boxed{v_1 = (1,1)^T}$$

For $\lambda_2 = -1$ eigenvector is:

$$(A + I)\underline{u} = \underline{0} \Leftrightarrow$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \boxed{v_2 = (1,-1)^T}$$

(b) Find a fundamental matrix solution $\Phi(t)$ to the system $\mathbf{x}' = A\mathbf{x}$.

$$\boxed{\Phi(t) = \begin{bmatrix} | & | \\ \underline{x}_1(t) & \underline{x}_2(t) \\ | & | \end{bmatrix}}$$ where $\underline{x}_1(t) = e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{x}_2(t) = e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\boxed{= \begin{bmatrix} e^{5t} & e^{-t} \\ e^{5t} & -e^{-t} \end{bmatrix}.}$$

(c) Find the matrix exponential e^{At}. Using your answer, find the solution to $\mathbf{x}' = A\mathbf{x}$, with $\mathbf{x}(0) = (1, 0)^T$.

$$\boxed{e^{At}} = \Phi(t)\,\Phi(0)^{-1} = \begin{bmatrix} e^{5t} & e^{-t} \\ e^{5t} & -e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

$$= -\frac{1}{2} \begin{bmatrix} e^{5t} & e^{-t} \\ e^{5t} & -e^{-t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boxed{= \frac{1}{2} \begin{bmatrix} e^{5t} + e^{-t} & e^{5t} - e^{-t} \\ e^{5t} - e^{-t} & e^{5t} + e^{-t} \end{bmatrix}}$$

(d) Find a particular solution $\mathbf{x}_p(t)$ to $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, where $\mathbf{f} = (4, 6)^T$. (**hint:** Use $\mathbf{x}_p(t) = \mathbf{u}$ = constant vector.)

$$\underline{0} = \underline{x}_p' = A\underline{x}_p + \underline{f} \Rightarrow \boxed{\underline{x}_p = -A^{-1}\underline{f}}$$

$$= -\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$= -\frac{1}{-5} \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} -10 \\ 0 \end{bmatrix} = \boxed{\begin{pmatrix} -2 \\ 0 \end{pmatrix}}$$

(e) Solve the initial value problem $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, with $\mathbf{x}(0) = (1, -1)^T$, and the same $\mathbf{f}$ as above.

$$\underline{x}(t) = \underline{x}_p(t) + \underline{x}_H(t) = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \Phi(t) \begin{pmatrix} a \\ b \end{pmatrix}$$

↑ could also use matrix exp.

at t = 0:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \Phi(0) \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \Phi(0)^{-1} \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

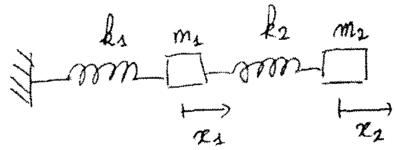
$$= \frac{-1}{2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

thus:

$$\boxed{\underline{x}(t) = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

Problem 2 (30 pts) Consider the two masses two springs system depicted below.



(a) Derive the second order system of DEs satisfied by $x_1(t)$ and $x_2(t)$ (displacements around the equilibrium position), assuming there are no external forces. Put the system in the form $M\mathbf{x}'' = K\mathbf{x}$, specifying what the matrices M and K are.

$$\begin{cases} m_1 x_1'' = -k_1 x_1 + k_2 (x_2 - x_1) \\ m_2 x_2'' = +k_2 (x_1 - x_2) \end{cases}$$

$$\Leftrightarrow \quad M\underline{x}'' = K\underline{x} \quad \text{where}$$

$$M = \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix}, \quad K = \begin{bmatrix} -(k_1+k_2) & k_2 \\ k_2 & -k_2 \end{bmatrix}.$$

(b) With $m_1 = 1$, $m_2 = 1/2$, $k_1 = 2$ and $k_2 = 1$ show that the system is equivalent to $\mathbf{x}'' = A\mathbf{x}$, with

$$A = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}.$$

$$\boxed{A = M^{-1}K = \begin{pmatrix} 1 & \\ & 2 \end{pmatrix} \begin{pmatrix} -2-1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix}.}$$

(c) Find the general solution to the system $\mathbf{x}'' = A\mathbf{x}$.

$$p(\lambda) = (-3-\lambda)(-2-\lambda) - 2 = \lambda^2 + 5\lambda + 4. \quad \lambda = \frac{-5 \pm \sqrt{25-16}}{2} = \frac{-5 \pm 3}{2}. \begin{cases} -1 \\ -4 \end{cases}$$

eigenvectors:

$\lambda_1 = -1$

$$(A+I)\underline{u} = \underline{0} \Leftrightarrow \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \boxed{\underline{v}_1 = (1, 2)^T.}$$

$\lambda_2 = -4$

$$(A+4I)\underline{u} = \underline{0} \Leftrightarrow \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \boxed{\underline{v}_2 = (1, -1)^T.}$$

$$\Rightarrow \underline{x}(t) = (a\cos\omega_1 t + b\sin\omega_1 t)\begin{pmatrix} 1 \\ 2 \end{pmatrix} + (c\cos\omega_2 t + d\sin\omega_2 t)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

where $\omega_1 = \sqrt{-\lambda_1} = 1$, $\omega_2 = \sqrt{-\lambda_2} = 2$.

(d) What are the natural frequencies of the system $\mathbf{x}'' = A\mathbf{x}$? Describe their corresponding modes of oscillation.

$\underline{\omega_1 = 1}$ angular freq: masses move in phase, with second mass oscillating with twice the amplitude as first one.

$\underline{\omega_2 = 2}$ masses move in opposite phases with same amplitude.

(e) Find a particular solution to the system $\mathbf{x}'' = A\mathbf{x} + \mathbf{f}$, where the external force is $\mathbf{f}(t) = (\cos\omega t, 0)^T$, and ω is not a resonance frequency of $\mathbf{x}'' = A\mathbf{x}$.

$\underline{x}_p(t) = \underline{a} \cos\omega t$. Plugging in the DE:

$$-\omega^2 \cos\omega t \, \underline{a} = \cos\omega t \, A \underline{a} + \cos\omega t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \quad (-\omega^2 \underline{I} + A)\, \underline{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \underline{a} = -\begin{bmatrix} -3+\omega^2 & 1 \\ 2 & -2+\omega^2 \end{bmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{-1}{p(-\omega^2)} \begin{bmatrix} -2+\omega^2 & -1 \\ -2 & -3+\omega^2 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where $p(\lambda) = \lambda^2 + 5\lambda + 4$
= char poly
$\neq 0$ when $\lambda \neq -\omega_i^2$
(not resonant freq).

$$= \frac{1}{p(-\omega^2)} \begin{bmatrix} 2-\omega^2 \\ 2 \end{bmatrix}$$

$$\Rightarrow \quad \boxed{\underline{x}_p(t) = \frac{\cos\omega t}{p(-\omega^2)} \begin{bmatrix} 2-\omega^2 \\ 2 \end{bmatrix}}$$

Problem 3 (10 pts) Consider the matrix

$$N = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) Show N is nilpotent.

$$N^2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N^3 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ <u>N is Nilpotent.</u>

(b) Use your answer to compute the matrix exponential e^{Nt}.

$$\boxed{e^{Nt} = I + tN + \frac{t^2}{2}N^2 = \begin{bmatrix} 1 & t & 2t + t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}}$$

Problem 4 (30 pts) Consider the two populations system,

$$\begin{cases} \dfrac{dx}{dt} = 3x - x^2 - \dfrac{1}{4}xy \\ \dfrac{dy}{dt} = xy - 2y. \end{cases}$$

(a) How does each population behave in the absence of the other one (logistic, exponential growth or decrease)? Describe the nature of the interaction between the $x-$population and the $y-$population (competition, cooperation, predation).

x- pop. is logistic when isolated from y-pop.

y- pop. exponentially decreases by itself.

moreover: y-pop benefits from increase in x-pop

x-pop ——— decrease in y-pop

$\Rightarrow$ predator / prey interaction where

x-pop = prey

y-pop = predator.

(b) Find the three critical points of the system (**hint:** one critical point is $(2, 4)^T$.)

$$\begin{cases} 3x - x^2 - \dfrac{xy}{4} = 0 \\ xy - 2y = 0 \end{cases} \Leftrightarrow \begin{cases} x(3 - x - y/4) = 0 \\ \text{and} \\ y(x-2) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0, \quad y = 0 \\ x = 2, \quad y = 4 \\ y = 0, \quad x = 3 \end{cases}$$

$\boxed{\text{Critical pts are } (0,0),\ (2,4) \text{ and } (3,0)}$

(c) Find the linearized system around the critical point $(2, 4)^T$.

$$J(x,y) = \begin{bmatrix} 3 - 2x - y/4 & -x/4 \\ y & x-2 \end{bmatrix}$$

$$J(2,4) = \begin{bmatrix} -2 & -1/2 \\ 4 & 0 \end{bmatrix}$$

$\Rightarrow$ linearized system around $(2,4)^T$ is:

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{bmatrix} -2 & -1/2 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{where } \begin{cases} x = 2 + u \\ y = 4 + v \end{cases}.$$

(d) Sketch the phase portrait for the linearized problem. (if the point is a saddle point, carefully plot the axis of the hyperbolic trajectories, if the point is a spiral, find the orientation by e.g. computing the tangent field at a couple of points). What type of critical point is this? Is it stable or unstable?

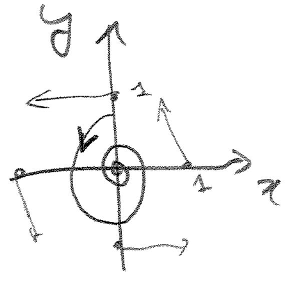
$$p(\lambda) = (-2-\lambda)(-\lambda) + 2 = \lambda^2 + 2\lambda + 2$$

char poly of $J(2,4)$

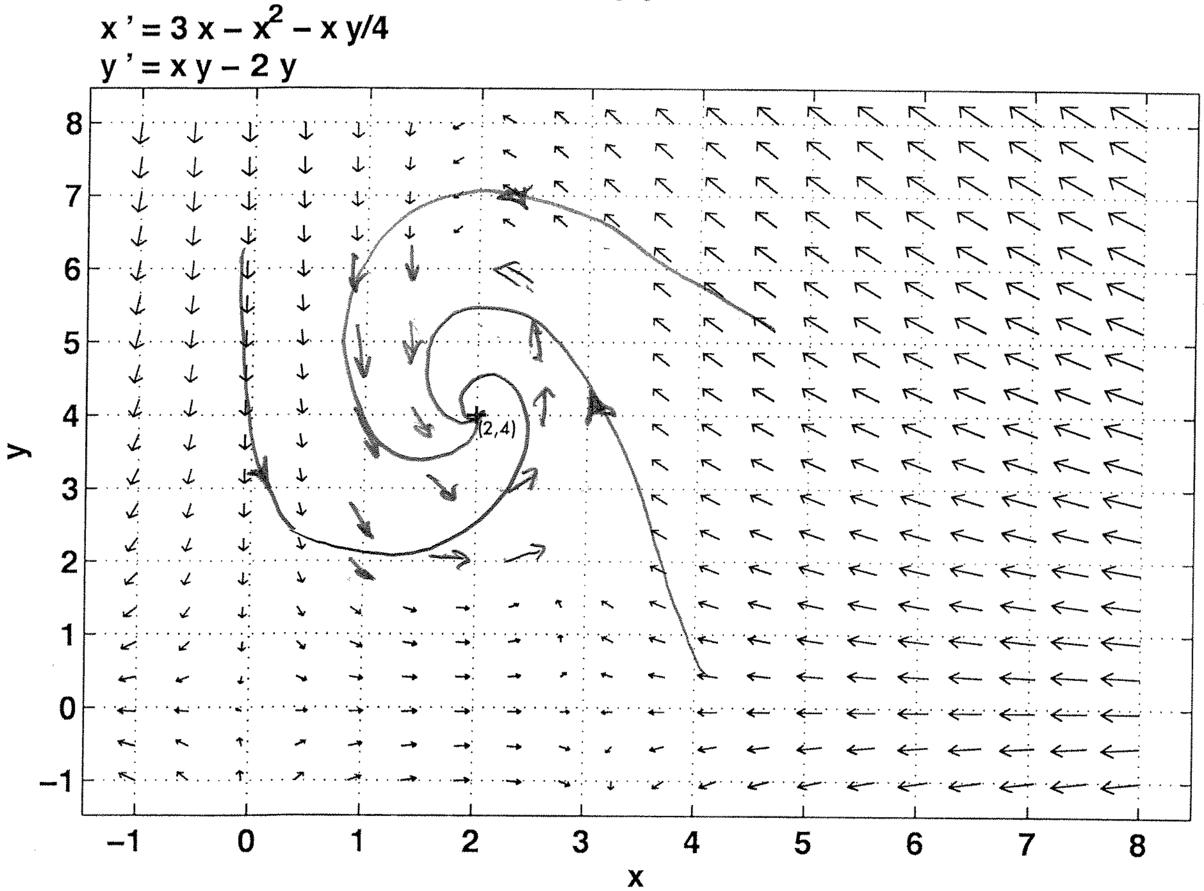
$$\lambda = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2}$$

$$= -1 \pm i$$

We have a complex conjugate pair of eigenvalues w/ negative real part $\Rightarrow$ $(0,0)$ is a stable spiral for lin. prob.



(e) Complete the phase portrait below around the critical point $(2,4)^T$. When $x(0) > 0$ and $y(0) > 0$, what do you expect about the x and y populations after a large time? Sketch a few trajectories representative of your predictions about the populations.



If the starting population is s.t. $x(0) > 0$ and $y(0) > 0$ then as $t \to \infty$ we expect the populations to converge to the stable critical pt $x = 2$, $y = 4$ as illustrated above.