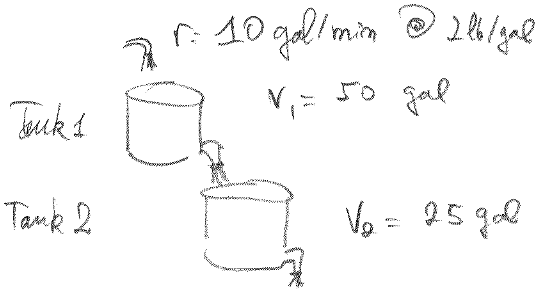
Math 2280-2, Practice Midterm Exam 2

March 27, 2008

Total: 100 points

Problem 1 (25 pts) Consider a two tank system with volumes $V_1 = 50$ gal and $V_2 = 25$ gal. A brine solution with salt concentration 2 lb/gal is pumped at a rate $r = 10$ gal/min into the first tank. The (perfectly mixed) solution flows from tank 1 to tank 2 and out of tank 2 with the same rate r. Initially at $t = 0$ min, the tanks contain fresh water. Let $x_i(t)$ denote the amount of salt (in pounds) in tank i at time t.

(a) Show that the system of DE's satisfied by $x_1(t)$ and $x_2(t)$ is of the form $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t)$, with $\mathbf{x}(t) = (x_1(t), x_2(t))^T$. Find A and $\mathbf{f}$.



Accounting for the quantity of salt entering and leaving the tanks per unit time, we get:

$$\begin{cases} \dfrac{dx_1}{dt} = 2r - r\dfrac{x_1}{V_1} = 20 - \dfrac{x_1}{5} \\[2ex] \dfrac{dx_2}{dt} = r\dfrac{x_1}{V_1} - r\dfrac{x_2}{V_2} = \dfrac{x_1}{5} - \dfrac{2x_2}{5} \end{cases}$$

Thus $\underline{x}' = A\underline{x} + \underline{f}$ where $A = \begin{bmatrix} -1/5 & 0 \\ 1/5 & -2/5 \end{bmatrix}$ and $\underline{f} = \begin{pmatrix} 20 \\ 0 \end{pmatrix}$.

(b) The homogeneous part of the system above is

$$\frac{dx_1}{dt} = -\frac{x_1}{5}$$
$$\frac{dx_2}{dt} = \frac{x_1}{5} - \frac{2x_2}{5} \tag{1}$$

Find a fundamental matrix solution $\Phi(t)$ to the homogeneous system (1).

We first find eigenvalues and eigenvectors of A:

$p(\lambda) = \det(A - \lambda I) = \left(-\frac{1}{5} - \lambda\right)\left(-\frac{2}{5} - \lambda\right) \Rightarrow \lambda_1 = -\frac{1}{5}$, $\lambda_2 = -\frac{2}{5}$ are eigenvalues.

Eigenvector:

$(A - \lambda_1)\underline{v_1} = \underline{0} \iff \begin{bmatrix} 0 & 0 \\ 1/5 & -1/5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$(A - \lambda_2)\underline{v_2} = \underline{0} \iff \begin{bmatrix} 1/5 & 0 \\ 1/5 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Rightarrow \quad \Phi(t) = \left[e^{-1/5t}\underline{v_1},\ e^{-2/5t}\underline{v_2} \right]$ is a fundamental matrix sol.

Note: $\Phi(0) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow \Phi(0)^{-1} = \frac{1}{1}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$.

(c) Find the matrix exponential e^{At}.

We use definition: $e^{At} = \Phi(t)\Phi(0)^{-1}$

$$= \begin{bmatrix} e^{-t/5} & 0 \\ e^{-t/5} & e^{-2t/5} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t/5} & 0 \\ e^{-t/5} - e^{-2t/5} & e^{-2t/5} \end{bmatrix}.$$

(d) Find a particular solution $\mathbf{x}_p(t)$ to (1) (hint: look for $\mathbf{x}_p(t) = \mathbf{u} =$ constant vector.)

Using hint: $\quad \underline{x}_p' = A\underline{x}_p + \underline{f}$

$\underline{x}_p = \text{const} \Rightarrow \quad \underline{0} = A\underline{x}_p + \underline{f}$

$$\Rightarrow \quad \underline{x}_p = -A^{-1}\underline{f}$$

$$= -\begin{bmatrix} -1/5 & 0 \\ 1/5 & -2/5 \end{bmatrix}^{-1} \begin{bmatrix} 20 \\ 0 \end{bmatrix}$$

$$= -\frac{25}{2}\begin{bmatrix} -2/5 & 0 \\ -1/5 & -1/5 \end{bmatrix} \begin{bmatrix} 20 \\ 0 \end{bmatrix}$$

$$= \frac{25}{2}\begin{bmatrix} 8 \\ 4 \end{bmatrix} = \boxed{\begin{bmatrix} 100 \\ 50 \end{bmatrix}}$$

(e) Find the quantity of salt at time t in each tank assuming at $t = 0$ the tanks contain only fresh water. What is the limit of $x_1(t)$ and $x_2(t)$ as $t \to \infty$?

$$\underline{x}(t) = \underline{x}_p(t) + \underbrace{\underline{x}_H(t)}_{= \Phi(t)\begin{pmatrix} a \\ b \end{pmatrix}}$$

$$\underbrace{\underline{x}(0)}_{= \underline{0}} = \underline{x}_p(0) + \Phi(0)\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \quad \begin{pmatrix} a \\ b \end{pmatrix} = -\Phi(0)^{-1}\underline{x}_p(0)$$

$$= \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 100 \\ 50 \end{pmatrix}$$

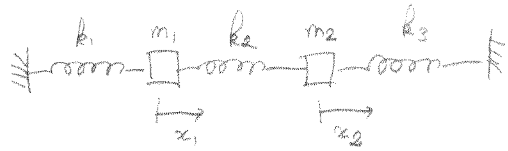
$$= \begin{pmatrix} -100 \\ 50 \end{pmatrix}$$

$$\Rightarrow \quad \underline{x}(t) = \begin{pmatrix} 100 \\ 50 \end{pmatrix} - 100\begin{pmatrix} e^{-t/5} \\ e^{-t/5} \end{pmatrix} + 50\begin{bmatrix} 0 \\ e^{-2t/5} \end{bmatrix}$$

We have:

$$\lim_{t\to\infty} \underline{x}(t) = \begin{pmatrix} 100 \\ 50 \end{pmatrix} = \underline{x}_p.$$

Problem 2 (25 pts) Consider the two masses three springs system depicted below.



(a) Derive the second order system satisfied by x_1 and x_2 (displacements around equilibrium position), assuming there are no external forces. Put the system in the from, $M\mathbf{x}'' = K\mathbf{x}$, specifying what the matrices M and K are.

Applying Newton's second law to each of the masses we get:

$$m_1 x_1'' = -k_1 x_1 + k_2 (x_2 - x_1)$$
$$m_2 x_2'' = -k_3 x_2 + k_2 (x_1 - x_2)$$

$$\Rightarrow \quad M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad K = \begin{bmatrix} -(k_1+k_2) & k_2 \\ k_2 & -(k_2+k_3) \end{bmatrix}$$

(b) With $m_1 = 1$, $m_2 = 2$, $k_1 = 1$ and $k_2 = k_3 = 2$ show that the system is equivalent to $\mathbf{x}'' = A\mathbf{x}$, with

$$A = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}.$$

From (a):

$$K = \begin{bmatrix} -3 & 2 \\ 2 & -4 \end{bmatrix}, \quad M = \begin{pmatrix} 1 & \\ & 2 \end{pmatrix} \Rightarrow A = M^{-1}K = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}$$

(c) Find the general solution to the system $\mathbf{x}'' = A\mathbf{x}$.

We need to first find the eigenvalues and eigenvectors of A.

$p(\lambda) = \det(A - \lambda I) = (-3-\lambda)(-2-\lambda) - 2 = \lambda^2 + 5\lambda + 4$

$\Rightarrow \lambda = \dfrac{-5 \pm \sqrt{25-16}}{2} \Rightarrow \begin{array}{l} \lambda_1 = -1 = -\omega_1^2 \\ \lambda_2 = -4 = -\omega_2^2 \end{array}$ $\quad \omega_i =$ natural freq.

The corresponding eigenvectors are:

$\underline{\lambda_1 = -1}$ $\quad (A + I)\underline{u} = 0 \Leftrightarrow \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underline{\lambda_2 = -4}$ $\quad (A + 4I)\underline{u} = \underline{0} \Leftrightarrow \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{v_2} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\Rightarrow \underline{x}(t) = (a \cos t + b \sin t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (c \cos 2t + d \sin 2t)\begin{pmatrix} 2 \\ -1 \end{pmatrix}.$

(d) What are the natural frequencies of the system $\mathbf{x}'' = A\mathbf{x}$? Describe their corresponding modes of oscillation.

The natural frequencies are:

$\underline{\omega_1 = 1}$: The two masses move in phase with equal amplitude.

$\underline{\omega_2 = 2}$: The two masses move in opposite phases with first mass moving with twice the amplitude of the second one.

(e) Find a particular solution to the system $\mathbf{x}'' = A\mathbf{x} + \mathbf{f}$, where the external force is $\mathbf{f}(t) = (\cos \omega t, 0)^T$, and ω is not a resonance frequency of $\mathbf{x}'' = A\mathbf{x}$.

We look for x_p of the form: $x_p(t) = \cos\omega t\, \underline{u}$, $\underline{u}$ = constant vector.

$$\Rightarrow \quad x_p'' = -\omega^2 \cos\omega t\, \underline{u} = \cos\omega t\, A\underline{u} + \cos\omega t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \underline{u} = -(A+\omega^2 I)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= -\begin{bmatrix} -3+\omega^2 & 2 \\ 1 & -2+\omega^2 \end{bmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\underbrace{\frac{1}{(\omega^2-4)(\omega^2-1)}}_{=1/p(-\omega^2)} \begin{bmatrix} -2+\omega^2 & -2 \\ -1 & -3+\omega^2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{(\omega^2-4)(\omega^2-1)} \begin{bmatrix} 2-\omega^2 \\ 1 \end{bmatrix}$$

$\neq 0$ since $\omega \neq$ resonance

$$\Rightarrow \quad x_p(t) = \frac{1}{(\omega^2-4)(\omega^2-1)} \begin{pmatrix} 2-\omega^2 \\ 1 \end{pmatrix} \cos\omega t$$

Problem 3 (10 pts) Consider the system $\mathbf{x}' = A\mathbf{x}$ with

$$A = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}.$$

This matrix has eigenvalues $\lambda_1 = 1+3i$, $\lambda_2 = 1-3i$ with associated eigenvectors $\mathbf{v}_1 = (1, i)^T$ and $\mathbf{v}_2 = (1, -i)^T$. Compute a real fundamental matrix solution for the system $\mathbf{x}' = A\mathbf{x}$.

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We simply take:

$$\underline{u}_1 = Re(e^{\lambda_1 t} \underline{v}_1) = Re\left((e^t(\cos 3t + i\sin 3t)(\underline{e}_1 + i\underline{e}_2)\right)$$

$$\underline{u}_2 = Im(e^{\lambda_1 t} \underline{v}_1) = Im(\quad " \quad)$$

$$\Rightarrow \quad \underline{u}_1(t) = e^t(\cos 3t\, \underline{e}_1 - \sin 3t\, \underline{e}_2) = e^t \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix}$$

$$\underline{u}_2(t) = e^t(\sin 3t\, \underline{e}_1 + \cos 3t\, \underline{e}_2) = e^t \begin{pmatrix} \sin 3t \\ +\cos 3t \end{pmatrix}$$

Problem 4 (15 pts) Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Compute e^{At}. (Hint: use that $A = 2I + N$, where N is nilpotent, or use chains of eigenvectors).

$$e^{At} = e^{(2I+N)t} = e^{2It} e^{Nt} = e^{2t}\left(I + \cancel{N}Nt + \frac{N^2}{2}t^2\right)$$

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad N^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow e^{At} = e^{2t} \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 5 (25 pts) A certain mass-spring system is modeled by the differential equation

$$x'' + 2x' + 20x - 5x^3 = 0 \tag{2}$$

(a) Write the equivalent first order system to (2), letting $y = x' =$ velocity.

$$\begin{cases} x' = y \\ y' = -20x - 2y + 5x^3 \end{cases}$$

note : this is prob 6.4.1[illegible]

(b) Find the critical points of this system (hint: $(0,0)$ and $(\pm 2, 0)$).

We need to find (x, y) s.t.

$$\begin{cases} y = 0 \\ \text{and} \\ -20x - 2y + 5x^3 = 0 \end{cases} \iff \begin{cases} y = 0 \\ \text{and} \\ x(x^2 - 4) = 0 \end{cases}$$

$\Rightarrow$ critical pts are $(0,0)$, $(2,0)$ and $(-2,0)$

(c) Find the linearized system around the critical point $(0,0)$.

We need to compute Jacobian at $(0,0)$:

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ -20 + 15x^2 & -2 \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ -20 & -2 \end{bmatrix}$$

If we let $x = 0 + u$
$y = 0 + v$

linearized system is:

$$\begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} = J(0,0) \begin{bmatrix} u \\ v \end{bmatrix}$$

(d) Sketch the phase portrait for the linearized problem. (if the point is a saddle point, carefully plot the axis of the hyperbolic trajectories, if the point is a spiral, find the direction of rotation by e.g. computing the tangent field at a couple of points).

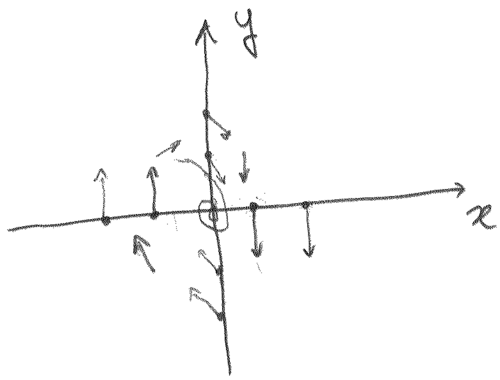
The only critical pt of the linearized problem is the origin $(0,0)$.
The eigenvalues of $J(0,0)$ are the roots of:

$$(-\lambda)(-2-\lambda)+20=0 \iff \lambda^2+2\lambda-20=0$$

$$\iff \lambda=\frac{-2\pm\sqrt{4-80}}{2}$$

$$=-1\pm i\sqrt{19}$$

$\Rightarrow$ Stable spiral pt.



orientation: (by sketching a few vectors from tangent field)

(e) Discuss the stability of the point $(0,0)$ for both the linearized problem and the original non-linear problem.

The Jacobian has a complex conjugate pair of eigenvalues with $\mathrm{Re}\,\lambda<0$. $\Rightarrow$ the origin is also a stable spiral point. Using theorem 6.2.2 (stability of almost linear systems)