

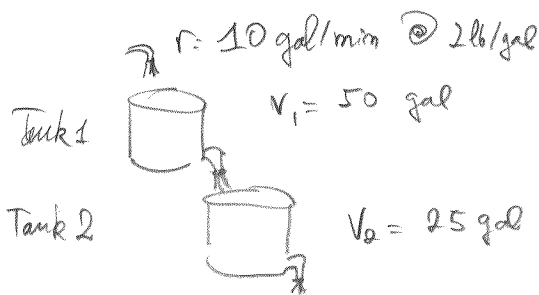
Math 2280-2, Practice Midterm Exam 2

March 27, 2008

Total: 100 points

Problem 1 (25 pts) Consider a two tank system with volumes $V_1 = 50$ gal and $V_2 = 25$ gal. A brine solution with salt concentration 2 lb/gal is pumped at a rate $r = 10$ gal/min into the first tank. The (perfectly mixed) solution flows from tank 1 to tank 2 and out of tank 2 with the same rate r . Initially at $t = 0$ min, the tanks contain fresh water. Let $x_i(t)$ denote the amount of salt (in pounds) in tank i at time t .

- (a) Show that the system of DE's satisfied by $x_1(t)$ and $x_2(t)$ is of the form $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t)$, with $\mathbf{x}(t) = (x_1(t), x_2(t))^T$. Find A and \mathbf{f} .



Accounting for the quantity of salt entering and leaving the tanks per unit time, we get:

$$\begin{cases} \frac{dx_1}{dt} = 2r - r \frac{x_1}{V_1} = 20 - \frac{x_1}{5} \\ \frac{dx_2}{dt} = r \frac{x_1}{V_1} - r \frac{x_2}{V_2} = \frac{x_1}{5} - \frac{2x_2}{5} \end{cases}$$

Thus $\underline{x}' = A\underline{x} + \underline{f}$ where $A = \begin{bmatrix} -1/5 & 0 \\ 1/5 & -2/5 \end{bmatrix}$ and $\underline{f} = \begin{pmatrix} 20 \\ 0 \end{pmatrix}$.

(b) The homogeneous part of the system above is

$$\begin{aligned} \frac{dx_1}{dt} &= -\frac{x_1}{5} \\ \frac{dx_2}{dt} &= \frac{x_1}{5} - \frac{2x_2}{5} \end{aligned} \quad (1)$$

Find a fundamental matrix solution $\Phi(t)$ to the homogeneous system (1).

We first find eigenvalues and eigenvectors of A :

$$p(\lambda) = \det(A - \lambda I) = \left(-\frac{1}{5} - \lambda\right)\left(-\frac{2}{5} - \lambda\right) \Rightarrow \lambda_1 = -\frac{1}{5} \quad \text{are eigenvalues.}$$

$$\lambda_2 = -\frac{2}{5}$$

Eigenvectors:

$$(A - \lambda_1)\underline{v}_1 = \underline{0} \Leftrightarrow \begin{bmatrix} 0 & 0 \\ 1/5 & -1/5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(A - \lambda_2)\underline{v}_2 = \underline{0} \Leftrightarrow \begin{bmatrix} 1/5 & 0 \\ 1/5 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \underline{\Phi}(t) = \left[e^{-1/5t} \underline{v}_1, e^{-2/5t} \underline{v}_2 \right] \quad \text{is a fundamental matrix sol.}$$

$$\underline{\text{Note:}} \quad \underline{\Phi}(0) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow \underline{\Phi}(0)^{-1} = \frac{1}{1} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

(c) Find the matrix exponential e^{At} .

$$\begin{aligned} \text{We use definition: } e^{At} &= \underline{\Phi}(t)\underline{\Phi}(0)^{-1} \\ &= \begin{bmatrix} e^{-t/5} & 0 \\ e^{-t/5} & e^{-2t/5} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{-t/5} & 0 \\ e^{-t/5} - e^{-2t/5} & e^{-2t/5} \end{bmatrix}. \end{aligned}$$

(d) Find a particular solution $\underline{x}_p(t)$ to (1) (hint: look for $\underline{x}_p(t) = \underline{u} = \text{constant vector}$.)

Using hint: $\underline{x}' = A\underline{x} + \underline{f}$

$$\underline{x}_p = \text{const} \Rightarrow \underline{0} = A\underline{x}_p + \underline{f}$$

$$\Rightarrow \underline{x}_p = -A^{-1}\underline{f}$$

$$= -\begin{bmatrix} -1/5 & 0 \\ 1/5 & -2/5 \end{bmatrix}^{-1} \begin{bmatrix} 20 \\ 0 \end{bmatrix}$$

$$= -\frac{25}{2} \begin{bmatrix} -2/5 & 0 \\ -1/5 & -1/5 \end{bmatrix} \begin{bmatrix} 20 \\ 0 \end{bmatrix}$$

$$= \frac{25}{2} \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 100 \\ 50 \end{bmatrix}$$

(e) Find the quantity of salt at time t in each tank assuming at $t = 0$ the tanks contain only fresh water. What is the limit of $x_1(t)$ and $x_2(t)$ as $t \rightarrow \infty$?

$$\underline{x}(t) = \underline{x}_p(t) + \underline{x}_h(t)$$

$$= \underline{\Phi}(t) \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\underline{x}(0) = \underline{x}_p(0) + \underline{\Phi}(0) \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = -\underline{\Phi}(0)^{-1} \underline{x}_p(0)$$

$$\begin{pmatrix} 100 \\ 50 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 100 \\ 50 \end{pmatrix}$$

$$= \begin{pmatrix} -100 \\ 50 \end{pmatrix}$$

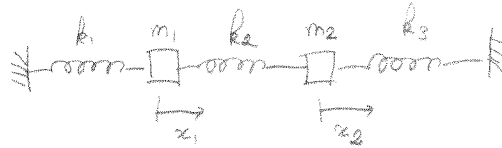
$$\Rightarrow \underline{x}(t) = \begin{pmatrix} 100 \\ 50 \end{pmatrix} - 100 \begin{pmatrix} e^{-t/5} \\ e^{-t/5} \end{pmatrix} + 50 \begin{pmatrix} 0 \\ e^{-2t/5} \end{pmatrix}$$

We have:

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$$\lim_{t \rightarrow \infty} \underline{x}(t) = \begin{pmatrix} 100 \\ 50 \end{pmatrix} = \underline{x}_p$$

Problem 2 (25 pts) Consider the two masses three springs system depicted below.



- (a) Derive the second order system satisfied by x_1 and x_2 (displacements around equilibrium position), assuming there are no external forces. Put the system in the form, $M\mathbf{x}'' = K\mathbf{x}$, specifying what the matrices M and K are.

Applying Newton's second law to each of the masses we get:

$$m_1 x_1'' = -k_1 x_1 + k_2 (x_2 - x_1)$$

$$m_2 x_2'' = -k_3 x_2 + k_2 (x_1 - x_2)$$

$$\Rightarrow M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad K = \begin{bmatrix} -(k_1 + k_2) & k_2 \\ k_2 & -(k_2 + k_3) \end{bmatrix}$$

- (b) With $m_1 = 1$, $m_2 = 2$, $k_1 = 1$ and $k_2 = k_3 = 2$ show that the system is equivalent to $\mathbf{x}'' = A\mathbf{x}$, with

$$A = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}.$$

From (a):

$$K = \begin{bmatrix} -3 & 2 \\ 2 & -4 \end{bmatrix}, \quad M = \begin{pmatrix} 1 & \\ & 2 \end{pmatrix} \Rightarrow A = M^{-1}K = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}$$

(c) Find the general solution to the system $\mathbf{x}'' = A\mathbf{x}$.

We need to first find the eigenvalues and eigenvectors of A .

$$p(\lambda) = \det(A - \lambda I) = (-3 - \lambda)(-2 - \lambda) - 2 = \lambda^2 + 5\lambda + 4$$

$$\Rightarrow \lambda = \frac{-5 \pm \sqrt{25 - 4 \cdot 4}}{2} \Rightarrow \begin{array}{l} \lambda_1 = -1 = -\omega_1^2 \\ \lambda_2 = -4 = -\omega_2^2 \end{array} \quad \omega_i = \text{natural freq.}$$

The corresponding eigenvectors are:

$$\lambda_1 = -1 \quad (A + I)\underline{u} = \underline{0} \Leftrightarrow \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -4 \quad (A + 4I)\underline{u} = \underline{0} \Leftrightarrow \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \underline{x}(t) = (a \cos t + b \sin t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (c \cos 2t + d \sin 2t) \begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdot$$

(d) What are the natural frequencies of the system $\mathbf{x}'' = A\mathbf{x}$? Describe their corresponding modes of oscillation.

The natural frequencies are:

$\omega_1 = 1$: The two masses move in phase with equal amplitude.

$\omega_2 = 2$: The two masses move in opposite phases with first mass moving with twice the amplitude of the second one.

- (e) Find a particular solution to the system $\mathbf{x}'' = A\mathbf{x} + \mathbf{f}$, where the external force is $\mathbf{f}(t) = (\cos \omega t, 0)^T$, and ω is not a resonance frequency of $\mathbf{x}'' = A\mathbf{x}$.

We look for \underline{x}_p of the form: $\underline{x}_p(t) = \cos \omega t \underline{u}$, $\underline{u} = \text{constant vector}$.

$$\Rightarrow \underline{x}_p'' = -\omega^2 \cos \omega t \underline{u} = \cos \omega t A \underline{u} + \cos \omega t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \Rightarrow \underline{u} &= -(A + \omega^2 I)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= - \begin{bmatrix} -3 + \omega^2 & 2 \\ 1 & -2 + \omega^2 \end{bmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = - \frac{1}{(\omega^2 - 4)(\omega^2 - 1)} \begin{bmatrix} -2 + \omega^2 & -2 \\ -1 & -3 + \omega^2 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{(\omega^2 - 4)(\omega^2 - 1)} \begin{bmatrix} 2 - \omega^2 \\ 1 \end{bmatrix} \end{aligned}$$

$= 1/p(-\omega^2)$
 $\neq 0$ since $\omega \neq \text{resonance}$

$$\Rightarrow \underline{x}_p(t) = \frac{1}{(\omega^2 - 4)(\omega^2 - 1)} \begin{pmatrix} 2 - \omega^2 \\ 1 \end{pmatrix} \cos \omega t$$

Problem 3 (10 pts) Consider the system $\mathbf{x}' = A\mathbf{x}$ with

$$A = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}.$$

This matrix has eigenvalues $\lambda_1 = 1 + 3i$, $\lambda_2 = 1 - 3i$ with associated eigenvectors $\mathbf{v}_1 = (1, i)^T$ and $\mathbf{v}_2 = (1, -i)^T$. Compute a real fundamental matrix solution for the system $\mathbf{x}' = A\mathbf{x}$.

We simply take: $\underline{u}_1 = \text{Re}(e^{\lambda_1 t} \underline{v}_1) = \text{Re}(e^t (\cos 3t + i \sin 3t) (\underline{e}_1 + i \underline{e}_2))$
 $\underline{u}_2 = \text{Im}(e^{\lambda_1 t} \underline{v}_1) = \text{Im}(e^t (\cos 3t + i \sin 3t) (\underline{e}_1 + i \underline{e}_2))$

$$\Rightarrow \underline{u}_1(t) = e^t (\cos 3t \underline{e}_1 - \sin 3t \underline{e}_2) = e^t \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix}$$

$$\underline{u}_2(t) = e^t (\sin 3t \underline{e}_1 + \cos 3t \underline{e}_2) = e^t \begin{pmatrix} \sin 3t \\ +\cos 3t \end{pmatrix}$$

Problem 4 (15 pts) Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Compute e^{At} . (Hint: use that $A = 2I + N$, where N is nilpotent, or use chains of eigenvectors).

$$e^{At} = e^{(2I+N)t} = e^{2It} e^{Nt} = e^{2t} \left(I + Nt + \frac{N^2 t^2}{2} \right)$$

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad N^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow e^{At} = e^{2t} \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 5 (25 pts) A certain mass-spring system is modeled by the differential equation

$$x'' + 2x' + 20x - 5x^3 = 0 \quad (2)$$

(a) Write the equivalent first order system to (2), letting $y = x'$ = velocity.

$$\begin{cases} x' = y \\ y' = -20x - 2y + 5x^3 \end{cases}$$

note: this is prob 6.4.13

(b) Find the critical points of this system (hint: $(0,0)$ and $(\pm 2,0)$).

We need to find (x,y) s.t.

$$\begin{cases} y=0 \\ \text{and} \\ -20x - 2y + 5x^3 = 0 \end{cases} \Leftrightarrow \begin{cases} y=0 \\ \text{and} \\ x(x^2 - 4) = 0 \end{cases}$$

\Rightarrow critical pts are $(0,0)$, $(2,0)$ and $(-2,0)$

(c) Find the linearized system around the critical point $(0,0)$.

We need to compute Jacobian at $(0,0)$:

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ -20 + 15x^2 & -2 \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ -20 & -2 \end{bmatrix}$$

If we let $x = 0 + u$
 $y = 0 + v$

linearized system is:

$$\begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} = J(0,0) \begin{bmatrix} u \\ v \end{bmatrix}$$

- (d) Sketch the phase portrait for the linearized problem. (if the point is a saddle point, carefully plot the axis of the hyperbolic trajectories, if the point is a spiral, find the direction of rotation by e.g. computing the tangent field at a couple of points).

The only critical pt of the linearized problem is the origin $(0,0)$.

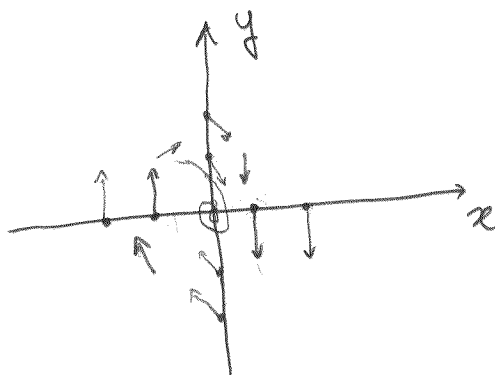
The eigenvalues of $J(0,0)$ are the roots of:

$$(-1)(-2-\lambda) + 20 = 0 \Leftrightarrow \lambda^2 + 2\lambda - 20 = 0$$

$$\Leftrightarrow \lambda = \frac{-2 \pm \sqrt{4 - 80}}{2}$$

$$= -1 \pm i\sqrt{19}$$

\Rightarrow Stable spiral pt.



orientation:

(by sketching a few vectors from tangent field)

- (e) Discuss the stability of the point $(0,0)$ for both the linearized problem and the original non-linear problem.

The Jacobian has a complex conjugate pair of eigenvalues with $\text{Re } \lambda < 0 \Rightarrow$ the origin is also a stable spiral point. Using theorem 6.2.2 (stability of almost linear systems)