# Math 2280-2, Practice Midterm Exam 2

March 27, 2008

**Total:** 100 points

**Problem 1 (25 pts)** Consider a two tank system with volumes $V_1 = 50$ gal and $V_2 = 25$ gal. A brine solution with salt concentration 2 lb/gal is pumped at a rate $r = 10$ gal/min into the first tank. The (perfectly mixed) solution flows from tank 1 to tank 2 and out of tank 2 with the same rate $r$. Initially at $t = 0$ min, the tanks contain fresh water. Let $x_i(t)$ denote the amount of salt (in pounds) in tank $i$ at time $t$.

(a) Show that the system of DE's satisfied by $x_1(t)$ and $x_2(t)$ is of the form $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t)$, with $\mathbf{x}(t) = (x_1(t), x_2(t))^T$. Find $A$ and $\mathbf{f}$.

(b) The homogeneous part of the system above is

$$\begin{aligned} \frac{dx_1}{dt} &= -\frac{x_1}{5} \\ \frac{dx_2}{dt} &= \frac{x_1}{5} - \frac{2x_2}{5} \end{aligned} \tag{1}$$

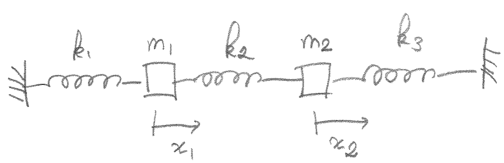
Find a fundamental matrix solution $\Phi(t)$ to the homogeneous system (1).

(c) Find the matrix exponential $e^{At}$.

(d) Find a particular solution $\mathbf{x}_p(t)$ to (1) (hint: look for $\mathbf{x}_p(t) = \mathbf{u} =$ constant vector.)

(e) Find the quantity of salt at time $t$ in each tank assuming at $t = 0$ the tanks contain only fresh water. What is the limit of $x_1(t)$ and $x_2(t)$ as $t \to \infty$?

**Problem 2 (25 pts)** Consider the two masses three springs system depicted below.




(a) Derive the second order system satisfied by $x_1$ and $x_2$ (displacements around equilibrium position), assuming there are no external forces. Put the system in the from, $M\mathbf{x}'' = K\mathbf{x}$, specifying what the matrices $M$ and $K$ are.

(b) With $m_1 = 1$, $m_2 = 2$, $k_1 = 1$ and $k_2 = k_3 = 2$ show that the system is equivalent to $\mathbf{x}'' = A\mathbf{x}$, with

$$A = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}.$$

(c) Find the general solution to the system $\mathbf{x}'' = A\mathbf{x}$.

(d) What are the natural frequencies of the system $\mathbf{x}'' = A\mathbf{x}$? Describe their corresponding modes of oscillation.

(e) Find a particular solution to the system $\mathbf{x}'' = A\mathbf{x} + \mathbf{f}$, where the external force is $\mathbf{f}(t) = (\cos \omega t, 0)^T$, and $\omega$ is not a resonance frequency of $\mathbf{x}'' = A\mathbf{x}$.

**Problem 3 (10 pts)** Consider the system $\mathbf{x}' = A\mathbf{x}$ with

$$A = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}.$$

This matrix has eigenvalues $\lambda_1 = 1+3i$, $\lambda_2 = 1-3i$ with associated eigenvectors $\mathbf{v}_1 = (1, i)^T$ and $\mathbf{v}_2 = (1, -i)^T$. Compute a real fundamental matrix solution for the system $\mathbf{x}' = A\mathbf{x}$.

**Problem 4 (15 pts)** Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Compute $e^{At}$. (Hint: use that $A = 2I + N$, where $N$ is nilpotent, or use chains of eigenvectors).

**Problem 5 (25 pts)** A certain mass-spring system is modeled by the differential equation

$$x'' + 2x' + 20x - 5x^3 = 0 \tag{2}$$

(a) Write the equivalent first order system to (2), letting $y = x' =$ velocity.

(b) Find the critical points of this system (hint: $(0, 0)$ and $(\pm 2, 0)$).

(c) Find the linearized system around the critical point $(0, 0)$.

(d) Sketch the phase portrait for the linearized problem. (if the point is a saddle point, carefully plot the axis of the hyperbolic trajectories, if the point is a spiral, find the direction of rotation by e.g. computing the tangent field at a couple of points).

(e) Discuss the stability of the point $(0, 0)$ for both the linearized problem and the original non-linear problem.