Math 2280-2, Midterm Exam 1

February 19, 2008

Total points: 110/100.

Problem 1 (25 pts) The fish population in a lake has a fertility rate $\beta = 10$, and a death rate of $\delta = 4 + P$. Fish are harvested from the lake at a constant rate of $h = 8$ fish per month.

(a) (4 pts) Show that the population of fish $P(t)$ satisfies the differential equation

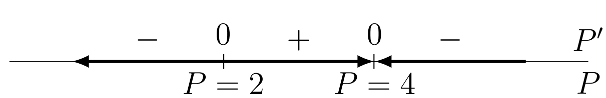
$$\frac{dP}{dt} = -P^2 + 6P - 8$$

The number of births per unit time is βP, and similarly the of deaths per unit time is δP. Thus the population of fish is governed by the differential equation

$$\frac{dP}{dt} = (\beta - \delta)P - h = (10 - 4 - P)P - 8 = -P^2 + 6P - 8.$$

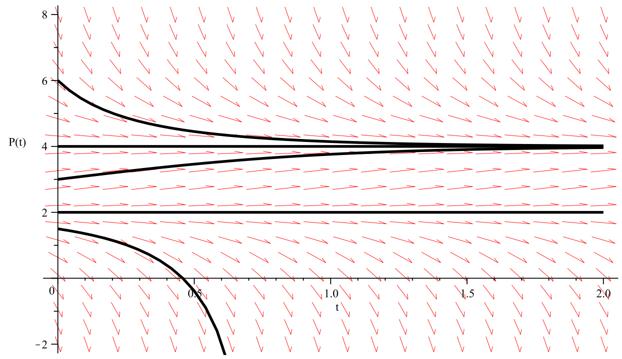
(b) (4 pts) Find the equilibrium solutions of the differential equation
We have $-P^2 + 6P - 8 = -(P-2)(P-4)$. Thus the equilibrium solutions of the differential equation are $P = 2$ and $P = 4$.

(c) (4 pts) Sketch the phase diagram for the differential equation. Identify the type of equilibrium solution.
The phase diagram for this DE is,



We see that $P = 2$ is unstable and $P = 4$ is stable.

(d) (4 pts) Sketch the slope field for the differential equation together with the equilibrium solutions and a few representative solutions (**note:** you do not need to find the formulas of the solutions to sketch them).



(e) (9 pts) Solve the differential equation with initial condition $P(0) = 3$. (Here you do need to find the formula for $P(t)$ for this initial condition). Check that your solution is consistent with the slope field, that is study $\lim_{t\to\infty} P(t)$.

We use separation of variables to solve the differential equation, which can be rewritten $dP/dT = -(P-2)(P-4)$.

$$\int \frac{dP}{(P-2)(P-4)} = -\int dt$$
$$\Rightarrow \frac{1}{2}\int \left(\frac{-1}{P-2} + \frac{1}{P-4}\right) dP = -t + C_1$$
$$\Rightarrow \ln\left|\frac{P-4}{P-2}\right| = -2t + C_2$$
$$\Rightarrow \frac{4-P}{P-2} = C_3 e^{-2t} \text{ since } P(0) = 3 \Rightarrow \frac{4-P}{P-2} > 0 \text{ for } t \text{ near zero.}$$

Using the initial condition in the last equation we get $C_3 = 1$. Solving for P gives,

$$P(t) = \frac{2e^{-2t} + 4}{e^{-2t} + 1}.$$

It is easy to check that $\lim_{t\to\infty} P(t) = 4$, which is consistent with $P = 4$ being a stable equilibrium.

Problem 2 (20 pts) Consider a 5 gallon tank, initially containing 1 gallon of pure water. A brine with concentration of 1 pound per gallon of salt enters the tank at 3 gallons per minute. The tank has a hole where the solution, assumed to be perfectly mixed, exits at a rate of 2 gallons per minute. Thus the tank volume is $V(t) = 1 + t$, and the tank overflows at $t = 4$ minutes.

(a) (5 pts) Show that the quantity of salt $x(t)$ (pounds) inside the tank before it overflows satisfies the differential equation,

$$\frac{dx}{dt} = 3 - 2\frac{x}{1+t}.$$

Using the notation from class we have that $r_i = 3$, $c_i = 1$, $r_o = 2$ and $V(t) = 1 + t$. The solution in the tank has a concentration $x(t)/V(t) = x/(1+t)$. Thus at any time before the overflow, the rate at which salt is entering the tank is

$$\frac{dx}{dt} = r_i c_i - r_o c_o = 3 - \frac{x}{1+t}.$$

(b) (15 pts) Use the integrating factor method to solve the above differential equation.
We rewrite the differential equation as,

$$\frac{dx}{dt} + \frac{2}{1+t}x = 3.$$

The integrating factor method tells us to multiply on both sides of the DE by

$$e^{\int 2dt/(1+t)} = e^{2\ln(1+t)} = (1+t)^2,$$

to obtain

$$\frac{d}{dt}((1+t)^2 x) = \frac{dx}{dt}(1+t)^2 + 2(1+t)x = 3(1+t)^2,$$

which by integration gives

$$(1+t)^2 x(t) = \int 3(1+t)^2 dt + C_1 = (1+t)^3 + C_1.$$

Thus the general form for x is,

$$x(t) = (1+t) + \frac{C_1}{(1+t)^2}.$$

Since there is no salt in the tank at $t = 0$ we get the value of the constant C_1

$$0 = x(0) = 1 + C_1 \quad \Rightarrow C_1 = -1.$$

and thus the answer

$$x(t) = 1 + t - \frac{1}{(1+t)^2}.$$

Problem 3 (25 pts) Consider the differential operator $L(y) = y^{(3)} - y'$.

(a) (5 pts) Write the characteristic polynomial $p(r)$ of L and find its roots.
The characteristic polynomial is $p(r) = r^3 - r = r(r^2 - 1)$. Its roots are $r_1 = 0$,$r_2 = 1$ and $r_3 = -1$.

(b) (5 pts) Find the general form of a solution y_H to the homogeneous differential equation $L(y) = 0$.

We have $y_H = a + be^x + ce^{-x}$.

(c) (10 pts) Find a particular solution y_p to $L(y) = e^{2x}$, using the undetermined coefficients method.

We look for a particular solution of the form $y_p(x) = Ae^{2x}$, thus

$$e^{2x} = L(y_p) = 8Ae^{2x} - 2Ae^{2x} \Rightarrow A = \frac{1}{6}.$$

Thus a particular solution is $y_p(x) = e^{2x}/6$.

(d) (5 pts) Solve the initial value problem for $L(y) = e^{2x}$ with initial conditions $y(0) = 1$, $y'(0) = 0$ and $y''(0) = 1$.

A general solution to $L(y) = e^{2x}$ can be written as

$$y = y_p + y_H = \frac{1}{6}e^{2x} + a + be^x + ce^{-x}.$$

It suffices to use the initial conditions to determine the constants. We obtain the following system of equations,

$$\begin{aligned} 1 = y(0) &= \frac{1}{6} + a + b + c \\ 0 = y'(0) &= \frac{1}{3} + b - c \\ 1 = y''(0) &= \frac{2}{3} + b + c \end{aligned}$$

Solving we get $b = 0$, $c = 1/3$ and $a = 1/2$. Thus the solution to the IVP is,

$$y(x) = \frac{1}{6}e^{2x} + \frac{1}{2} + \frac{1}{3}e^{-x}.$$

Problem 4 (30 pts) Consider a mass-spring system with damping, where the mass $m = 1$, the spring constant $k = 2$ and the damping constant $c = 3$. Recall that the displacement $x(t)$ from the equilibrium position satisfies the differential equation

$$mx'' + cx' + kx = 0.$$

(a) (5 pts) Find the characteristic polynomial $p(r)$ of this differential equation and its roots for the given values of m, k and c.

The characteristic polynomial of the DE is $p(r) = r^2 + 3r + 2$, its roots are $r_1 = -1$ and $r_2 = -2$.

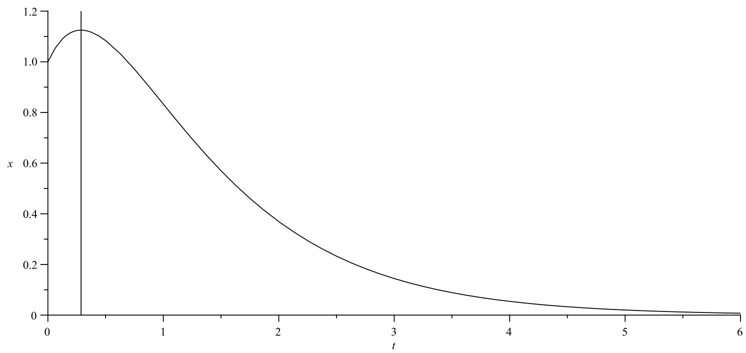
(b) (5 pts) What kind of motion is this?

The discriminant above is positive $\Delta^2 = c^2 - 4k = 9 - 8 = 1 > 0$. Thus the motion is **overdamped**.

(c) (5 pts) Find the general form of a solution to the differential equation.

A solution to the DE has the form $x(t) = ae^{-t} + be^{-2t}$.

(d) (5 pts) Sketch (qualitatively) a typical $x(t)$ for this system.



(e) (5 pts) Solve the initial value problem for this DE with $x(0) = 1$ and $x'(0) = 1$.

We need to find the constants a and b so that $x(t)$ satisfies the initial conditions. We get the system of equations,

$$1 = x(0) = a + b$$
$$1 = x'(0) = -a - 2b$$

which gives $b = -2$ and $a = 3$. Therefore $x(t) = 3e^{-t} - 2e^{-2t}$.

(f) (5 pts) For these particular initial conditions, what is the farthest position from the equilibrium that the mass reaches? (**hint:** find T such that $x'(T) = 0$).

We need to find T for which $0 = x'(T) = -3e^{-T} + 4e^{-2T}$, thus $T = \ln(4/3) \approx 0.28$. Evaluating $x(T) = 3e^{-\ln(4/3)} - 2e^{-2\ln(4/3)} = \frac{9}{4} - \frac{9}{8} = \frac{9}{8} = 1.125$.

Problem 5 (10 pts) Consider the initial value problem $y' = f(x, y)$, $y(0) = y_0$.

(a) Write down the (unimproved) Euler's method, to approximate the solution $y(x)$ to the IVP above on the interval $[x_0, x_n] = [0, 1]$, at the points $x_i = ih$, $i = 0 \ldots n$ with $h = 1/n$.

Euler's method
for $i = 0 \ldots n - 1$
  $y_{i+1} \leftarrow y_i + hf(x_i, y_i)$
end for

(b) What is the order of accuracy that you can expect with Euler's method?

Euler's method is of order 1.