Math 2280-2, Practice Midterm Exam 1

February 12, 2008

Total: 110/100 points

Problem 1 (25 pts) A certain population P can be modeled by the differential equation

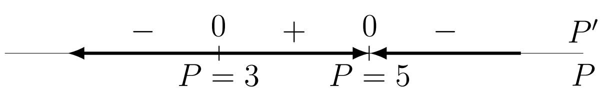
$$\frac{dP}{dt} = -P^2 + 8P - 15$$

(a) (4 pts) Find the equilibrium solutions of the differential equation

We have $-P^2 + 8P - 15 = -(P-3)(P-5)$. Thus the equilibrium solutions of the differential equation are $P = 3$ and $P = 5$.

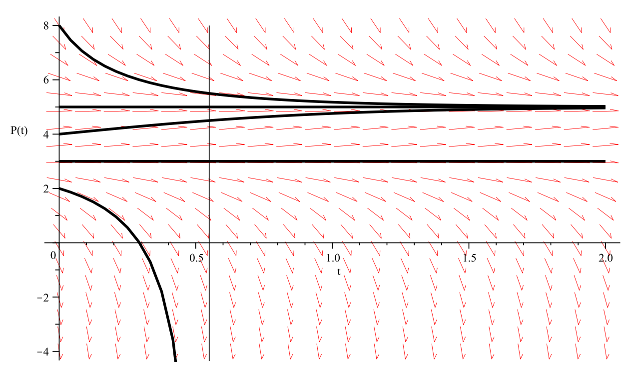
(b) (4 pts) Sketch the phase diagram for the differential equation. Identify the type of equilibrium solution.

The phase diagram for this DE is,



We see that $P = 3$ is unstable and $P = 5$ is stable.

(c) (4 pts) Sketch the slope field for the differential equation together with the equilibrium solutions and a few representative solutions (**note:** you do not need to find the formulas of the solutions to sketch them).



(d) (9 pts) Solve the differential equation with initial condition $P(0) = 2$. (Here you do need to find the formula for $P(t)$ for this initial condition).

We rewrite the differential as $dP/dt = -(P-3)(P-5)$ and use separation of variables:

$$\begin{aligned}
&\int \frac{dP}{(P-3)(P-5)} = -\int dt \\
\Rightarrow &\frac{1}{2}\int \left(\frac{-1}{P-3} + \frac{1}{P-5}\right) dP = -t + C_1 \\
\Rightarrow &\ln\left|\frac{P-5}{P-3}\right| = -2t + C_2 \\
\Rightarrow &\frac{P-5}{P-3} = C_3 e^{-2t} \quad \text{since } P(0) = 2 \;\Rightarrow\; \frac{P-5}{P-3} > 0 \text{ for } t \text{ near zero.}
\end{aligned}$$

Using the initial condition in the last equation we get $C_3 = 3$. Solving for P gives,

$$P(t) = \frac{9e^{-2t} - 5}{3e^{-2t} - 1}.$$

This is plotted in the slope field above.

(e) (4 pts) Check that your solution is consistent with the slope field, that is study the behavior of $P(t)$. In particular, find the value T for which $\lim_{t \to T} P(t) = -\infty$.

The denominator of $P(t)$ can be zero. This happens for T such that $3e^{-2T} = 1$, i.e. for $T = \ln(3)/2$. This is the vertical asymptote plotted in the slope field and is consistent with the behavior of $P(t)$.

Problem 2 (25 pts) Consider a 10 gallon tank, initially full of pure water. A brine with concentration of 1 pound of salt per gallon enters the tank at the rate of 1 gallon per minute. The mixture at the tank is kept perfectly mixed, and the tank has a hole that lets the solution escape at 2 gallons per minute. Thus the tank will be empty after exactly 10 minutes and the volume of solution in the tank is $V(t) = 10 - t$.

(a) (5 pts) Show that the quantity of salt $x(t)$ (pounds) inside the tank before it is empty satisfies the differential equation,

$$\frac{dx}{dt} = 1 - \frac{2}{10-t}x.$$

Using the notation from class we have that $r_i = 1$, $c_i = 1$ and $r_o = 2$. The concentration of salt in the tank is $c_o = x/V = x/(10-t)$. Thus at any time $0 \le t \le 10$, the rate at which the quantity of salt in the tank changes is

$$\frac{dx}{dt} = r_i c_i - r_o c_o = 1 - \frac{2}{10-t}x.$$

(b) (15 pts) Use the integrating factor method to solve the above differential equation.

We rewrite the differential equation as,

$$\frac{dx}{dt} + \frac{2}{10-t}x = 1.$$

The integrating factor method tells us to multiply on both sides of the DE by

$$e^{\int 2dt/(10-t)} = e^{-2\ln(10-t)} = \frac{1}{(10-t)^2}, \quad \text{(Note the minus sign in the exponential)}$$

to obtain

$$\frac{d}{dt}\left(\frac{x}{(10-t)^2}\right) = \frac{dx}{dt}\frac{1}{(10-t)^2} + \frac{2}{(10-t)^3}x = \frac{1}{(10-t)^2},$$

which by integration gives

$$\frac{x(t)}{(10-t)^2} = \int \frac{dt}{(10-t)^2} + C_1 = \frac{1}{10-t} + C_1.$$

Thus the general form for x is,

$$x(t) = (10-t) + C_1(10-t)^2.$$

Since there is no salt in the tank at $t = 0$ we get the value of the constant C_1

$$0 = x(0) = 10 + 100C_1 \quad \Rightarrow C_1 = -1/10.$$

and thus the answer

$$x(t) = (10-t) - \frac{1}{10}(10-t)^2.$$

(c) (5 pts) What is the maximum quantity of salt ever in the tank?

The maximum amount salt occurs at T where $x'(T) = 0$. Using the expression we have found for $x(t)$ we get

$$x'(t) = -1 + \frac{1}{5}(10-t).$$

Setting $x'(T) = 0$ we get $T = 5$. Evaluating, the maximum quantity of salt in the tank is $x(T) = 5/2 = 2.5$ pounds.

Problem 3 (25 pts) Consider the differential operator $L(y) = y'' + 2y' + 2y = 0$.

(a) (5 pts) Write the characteristic polynomial $p(r)$ of L and find its roots.

The characteristic polynomial is $p(r) = r^2 + 2r + 2 = (r+1+i)(r+1-i)$ Its roots are complex: $r_1 = -1+i$ and $r_2 = -1-i$.

(b) (5 pts) Find the general form of a solution y_H to the homogeneous differential equation $L(y) = 0$.

A general solution has the form $y_H(x) = Ae^{(-1+i)x} + Be^{(-1-i)x} = e^{-x}(a\cos(x) + b\sin(x))$.

(c) (10 pts) Find a particular solution y_p to $L(y) = \cos(2x)$, using the method of undetermined coefficients.

Our search space for y_p is $V = \{\cos 2x, \sin 2x\}$, that is we seek a particular solution of the form $y_p = a\cos 2x + b\sin 2x$. We now find the matrix $(L)_{\mathcal{B}}$ of L in the basis $\mathcal{B} = \{\cos 2x, \sin 2x\}$ of V. This is the matrix with columns being the images of the basis vectors:

$$\begin{aligned} L(\cos 2x) &= -4\cos 2x - 4\sin 2x + 2\cos 2x = -2\cos 2x - 4\sin 2x, \text{ and} \\ L(\sin 2x) &= -4\sin 2x + 4\cos 2x + 2\sin 2x = 4\cos 2x - 2\sin 2x. \end{aligned}$$

Then we get the linear system $\mathbf{Ac} = \mathbf{b}$, with

$$\mathbf{A} = (L)_{\mathcal{B}} = \begin{bmatrix} -2 & 4 \\ -4 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = (\cos 2x)_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solving the system we get

$$\mathbf{c} = \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{20}\begin{bmatrix} -2 & -4 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -1 \\ 2 \end{bmatrix},$$

thus a particular solution to $L(y) = \cos 2x$ is

$$y_p(x) = -\frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x.$$

Problem 4 (25 pts) Consider a mass-spring system with damping, where the mass $m = 1$, the string constant $k = 104$ and the damping constant $c = 4$. Recall that the displacement $x(t)$ from the equilibrium position satisfies the differential equation

$$mx'' + cx' + kx = 0.$$

(a) (5 pts) Find the characteristic polynomial $p(r)$ of this DE and its roots for the given values of m, k and c.

The characteristic polynomial of the DE is $p(r) = r^2 + 4r + 104$, the roots are complex $r_1 = -2 + 10i$ and $r_2 = -2 - 10i$.

(b) (5 pts) What kind of motion is this?

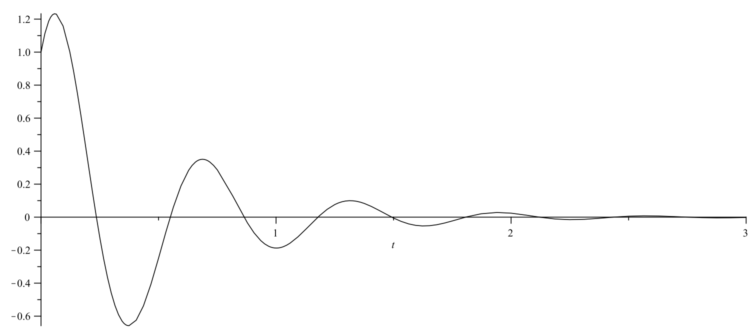
Since the discriminant we found above is $\Delta^2 = c^2 - 4k = 16 - 416 = -400 < 0$, we have an **underdamped** motion.

(c) (5 pts) Find the general form of a solution to the DE.

A general solution to the DE has the form

$$x(t) = Ae^{(-2+10i)t} + Be^{(-2-10i)t}, \quad \text{or equivalently} \quad x(t) = e^{-2t}(a\cos 10t + b\sin 10t).$$

(d) (5 pts) Sketch (qualitatively) a typical $x(t)$ for this system.



(e) (5 pts) What is the pseudoperiod of the motion?

The function $e^{2t}x(t)$ is $\frac{2\pi}{10}$−periodic. This is also the pseudoperiod of the motion.

Problem 5 (10 pts) Let L be a third order *linear* differential operator. Assume y_1, y_2, and y_3 are solutions to the following initial value problems,

$$L(y_1) = 0,\ y_1(0) = 1,\ y_1'(0) = 0,\ y_1''(0) = 0 \tag{1}$$
$$L(y_2) = 0,\ y_2(0) = 0,\ y_2'(0) = 1,\ y_2''(0) = 0 \tag{2}$$
$$L(y_3) = 0,\ y_3(0) = 0,\ y_3'(0) = 0,\ y_3''(0) = 1 \tag{3}$$

Let a, b and c be some real constants. Give a solution y to the following IVP

$$L(y) = 0,\ y(0) = a,\ y'(0) = b,\ y''(0) = c.$$

What result from class did you use?

By the superposition principle we that $y = ay_1 + by_2 + cy_3$ solves the IVP. This is a translation of the linearity of L and of the derivative operation,

$$\begin{aligned} L(y) &= L(ay_1 + by_2 + cy_3) = aL(y_1) + bL(y_2) + cL(y_3) = 0 \\ y(0) &= ay_1(0) + by_2(0) + cy_3(0) = a \\ y'(0) &= ay_1'(0) + by_2'(0) + cy_3'(0) = b \\ y''(0) &= ay_1''(0) + by_2''(0) + cy_3''(0) = c. \end{aligned}$$

Note: this proof was not required. Invoking the superposition principle is enough.