Math 2280-2, Practice Midterm Exam 1

February 12, 2008

Total: 110/100 points

Problem 1 (25 pts) A certain population P can be modeled by the differential equation

$$\frac{dP}{dt} = -P^2 + 8P - 15$$

(a) (4 pts) Find the equilibrium solutions of the differential equation

(b) (4 pts) Sketch the phase diagram for the differential equation. Identify the type of equilibrium solution.

(c) (4 pts) Sketch the slope field for the differential equation together with the equilibrium solutions and a few representative solutions (**note:** you do not need to find the formulas of the solutions to sketch them).

(d) (9 pts) Solve the differential equation with initial condition $P(0) = 2$. (Here you do need to find the formula for $P(t)$ for this initial condition).

(e) (4 pts) Check that your solution is consistent with the slope field, that is study the behavior of $P(t)$. In particular, find the value T for which $\lim_{t \to T} P(t) = -\infty$.

Problem 2 (25 pts) Consider a 10 gallon tank, initially full of pure water. A brine with concentration of 1 pound of salt per gallon enters the tank at the rate of 1 gallon per minute. The mixture at the tank is kept perfectly mixed, and the tank has a hole that lets the solution escape at 2 gallons per minute. Thus the tank will be empty after exactly 10 minutes and the volume of solution in the tank is $V(t) = 10 - t$.

(a) (5 pts) Show that the quantity of salt $x(t)$ (pounds) inside the tank before it is empty satisfies the differential equation,

$$\frac{dx}{dt} = 1 - \frac{2}{10 - t}x.$$

(b) (15 pts) Use the integrating factor method to solve the above differential equation.

(c) (5 pts) What is the maximum quantity of salt ever in the tank?

Problem 3 (25 pts) Consider the differential operator $L(y) = y'' + 2y' + 2y = 0$.

(a) (5 pts) Write the characteristic polynomial $p(r)$ of L and find its roots.

(b) (5 pts) Find the general form of a solution y_H to the homogeneous differential equation $L(y) = 0$.

(c) (10 pts) Find a particular solution y_p to $L(y) = \cos(2x)$, using the method of undetermined coefficients.

Problem 4 (25 pts) Consider a mass-spring system with damping, where the mass $m = 1$, the string constant $k = 104$ and the damping constant $c = 4$. Recall that the displacement $x(t)$ from the equilibrium position satisfies the differential equation

$$mx'' + cx' + kx = 0.$$

(a) (5 pts) Find the characteristic polynomial $p(r)$ of this DE and its roots for the given values of m, k and c.

(b) (5 pts) What kind of motion is this?

(c) (5 pts) Find the general form of a solution to the DE.

(d) (5 pts) Sketch (qualitatively) a typical $x(t)$ for this system.

(e) (5 pts) What is the pseudoperiod of the motion?

Problem 5 (10 pts) Let L be a third order *linear* differential operator. Assume y_1, y_2, and y_3 are solutions to the following initial value problems,

$$L(y_1) = 0,\ y_1(0) = 1,\ y_1'(0) = 0,\ y_1''(0) = 0 \tag{1}$$
$$L(y_2) = 0,\ y_2(0) = 0,\ y_2'(0) = 1,\ y_2''(0) = 0 \tag{2}$$
$$L(y_3) = 0,\ y_3(0) = 0,\ y_3'(0) = 0,\ y_3''(0) = 1 \tag{3}$$

Let a, b and c be some real constants. Give a solution y to the following IVP

$$L(y) = 0,\ y(0) = a,\ y'(0) = b,\ y''(0) = c.$$

What result from class did you use?