Math 2280-2
Problem 4.3.10

Invoke some packages that are needed

```
> with(DEtools):
```

Define the slope function

```
> f := (t,x) -> Matrix([[1., 2.],[1., 0.]]).x + Vector([0.,exp(-t)
  ]);
```

$$f := (t, x) \rightarrow \begin{bmatrix} 1. & 2. \\ 1. & 0. \end{bmatrix} . x + \mathit{Vector}\left(\left[0., e^{-t}\right]\right) \tag{1}$$

```
> xtrue:= t-> (1/9)*Vector([2*exp(2*t) - 2*exp(-t) - 6*t*exp(-t),
  exp(2*t) - exp(-t) + 6*t*exp(-t)]);
```

$$\mathit{xtrue} := t \rightarrow \frac{1}{9}\ \mathit{Vector}\left(\left[2\,e^{2\,t} - 2\,e^{-t} - 6\,t\,e^{-t}, e^{2\,t} - e^{-t} + 6\,t\,e^{-t}\right]\right) \tag{2}$$

Initial condition

```
> x0:=Vector([0,0]);
```

$$\mathit{x0} := \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3}$$

Calculation with n=10

Setting up some data structures for the iterations

```
> t0:=0.; tn:=1.; n :=10; h := (tn-t0)/n;
  xvals := Vector(n+1): tvals := Vector(n+1):
  xvals[1]:=x0: tvals[1]:=t0:
```

$$t0 := 0.$$

$$tn := 1.$$

$$n := 10$$

$$h := 0.1000000000 \tag{1.1}$$

RK4 using vectors

```
> for i from 1 to n do
    x := xvals[i]: t := tvals[i]:
     k1:= f(t,x):                  # left hand slope
     k2:= f(t+h/2,x+h*k1/2):       # midpoint slope: first
  approx.
     k3:= f(t+h/2,x+h*k2/2):       # midpoint slope: second
  approx.
     k4:= f(t+h,x+h*k3):           # right hand slope approx.
     k:=(k1+2*k2+2*k3+k4)/6:       # Simpson's integration rule
     xvals[i+1]:= x + h*k:         # improved Euler update
     tvals[i+1]:= t+h:             # increase x
  end do:
```

This is the approximation to x(1)

```
> print(tvals[n+1],xvals[n+1],xtrue(tvals[n+1]));
```

$$1.000000000, \begin{bmatrix} 1.31497663978267809 \\ 1.02536729235822310 \end{bmatrix}, \begin{bmatrix} 1.31500851888888870 \\ 1.02538370055555550 \end{bmatrix} \tag{1.2}$$

Calculation with n=20

Setting up some data structures for the iterations

```
> t0:=0.; tn:=1.; n :=20; h := (tn-t0)/n;
  xvals := Vector(n+1): tvals := Vector(n+1):
  xvals[1]:=x0: tvals[1]:=t0:
```

$$t0 := 0.$$

$$tn := 1.$$

$$n := 20$$

$$h := 0.05000000000 \tag{2.1}$$

RK4 using vectors

```
> for i from 1 to n do
   x := xvals[i]: t := tvals[i]:
    k1:= f(t,x):                  # left hand slope
    k2:= f(t+h/2,x+h*k1/2):       # midpoint slope: first
  approx.
    k3:= f(t+h/2,x+h*k2/2):       # midpoint slope: second
  approx.
    k4:= f(t+h,x+h*k3):           # right hand slope approx.
    k:=(k1+2*k2+2*k3+k4)/6:       # Simpson's integration rule
    xvals[i+1]:= x + h*k:         # improved Euler update
    tvals[i+1]:= t+h:             # increase x
  end do:
```

This is the approximation to x(1)

```
> print(tvals[n+1],xvals[n+1],xtrue(tvals[n+1]));
```

$$1.000000000, \begin{bmatrix} 1.31500633616826956 \\ 1.02538258147847982 \end{bmatrix}, \begin{bmatrix} 1.31500851888888870 \\ 1.02538370055555550 \end{bmatrix} \tag{2.2}$$

Calculation of the true solution using Maple

This part of the code confirms there is a typo in the books xtrue.

```
> unassign('x'); unassign('t');
  sys:={diff(x(t),t)=x(t)+2*y(t), diff(y(t),t)=x(t)+exp(-t)};
>
```

$$sys := \left\{ \frac{d}{dt} x(t) = x(t) + 2\, y(t), \frac{d}{dt} y(t) = x(t) + e^{-t} \right\} \tag{3.1}$$

```
> dsolve(sys union {x(0)=0,y(0)=0});
```

$$\left\{ x(t) = \frac{2}{9} e^{2t} - \frac{2}{9} e^{-t} - \frac{2}{3} t e^{-t}, y(t) = \frac{1}{9} e^{2t} - \frac{1}{9} e^{-t} + \frac{2}{3} t e^{-t} \right\} \tag{3.2}$$

```
> xtrue(t)[1]- (-(2/9)*exp(-t)+(2/9)*exp(2*t)-(2/3)*t*exp(-t));
```

$$0 \tag{3.3}$$

```
> xtrue(t)[2] - (-(1/9)*exp(-t)+(1/9)*exp(2*t)+(2/3)*t*exp(-t))
  ;
```