Mock final exam
Math 2210-3 fall 2007

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1. Consider the curve

$$\mathbf{r}(t) = t\mathbf{i} + \sqrt{10t}\mathbf{j} + \sqrt{25 - 10t - t^2}\mathbf{k}, \quad 0 \le t \le 2.$$

(a) Show that the curve lies on a sphere centered at the origin.

(b) Where does the tangent line at $t_0 = 2$ interesects the yz plane?

Answer:

(a) We simply need to show that the distance $\mathbf{r}(t)$ to the origin is constant. Thus we write,

$$\|\mathbf{r}(t) - \mathbf{0}\|^2 = t^2 + 10t + 25 - 10t - t^2 = 25$$

which means that $\|\mathbf{r}(t)\| = 5$, or equivalently that the curve lies in the sphere of radius 5 centered at the origin.

(b) The tangent line at t_0 passes through t_0 and has direction vector $\mathbf{r}'(t_0)$, therefore its parametric equation is

$$\mathbf{T}(t) = \mathbf{r}(t_0) + (t - t_0)\mathbf{r}'(t_0).$$

Plugging in the expression for $\mathbf{r}(t)$ we get $\mathbf{r}(2) = (2, 2\sqrt{5}, 1)$. Moreover we have,

$$\mathbf{r}'(t) = \left(1, \frac{10}{2\sqrt{10t}}, \frac{-10 - 2t}{2\sqrt{25 - 10t - t^2}}\right),$$

thus evaluating at $t_0 = 2$, $\mathbf{r}'(2) = (1, \sqrt{5}/2, -7)$, and the tangent line at $t = t_0$ has the parametric equation,

$$\mathbf{T}(t) = \begin{pmatrix} 2 + (t - 2) \\ 2\sqrt{5} + \frac{\sqrt{5}}{2}(t - 2) \\ 1 - 7(t - 2) \end{pmatrix}.$$

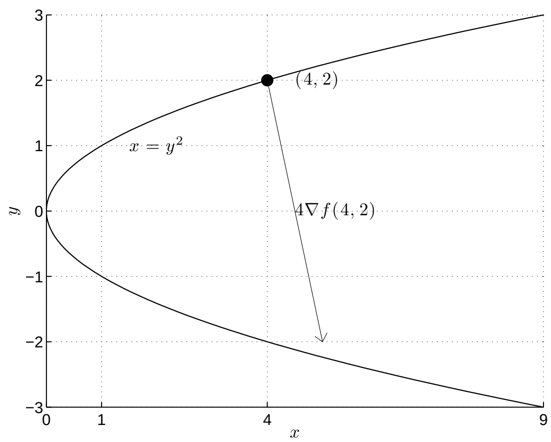
Now we note that points in the yz plane satisfy the equation $x = 0$, thus to find the intersection we must have that the x component of $\mathbf{T}(t)$ is zero. This means $t = 0$, and thus the intersection of the tangent line to the yz plane is $\mathbf{T}(0) = (0, \sqrt{5}, 15)$.

2. Sketch the level curve of $f(x, y) = x/y^2$ that goes through the point $\mathbf{p} = (4, 2)$. Calculate $\nabla f(\mathbf{p})$ and draw the vector $4\nabla f(\mathbf{p})$, placing its initial point at $\mathbf{p}$. What should be true about $\nabla f(\mathbf{p})$?

   Answer:

   The level curves are the curves satisfying $f(x, y) = k$. Here the level curves are of the form $x = ky^2$ (parabolas). For $\mathbf{p} = (4, 2)$, we have $f(\mathbf{p}) = 1$, therefore the level curve that passes through $\mathbf{p}$ is given by the equation $x = y^2$ (see figure 1). The gradient $\nabla f(\mathbf{p}) = (1/4, -1)$ should be perpendicular to the (tangent of the) level curve at $\mathbf{p}$ as can be seen in figure 1.

Figure 1 Level curve of $f(x, y) = x/y^2$ passing through $(4, 2)$.



3. A particle moves according to

$$\mathbf{r}(t) = (t + 1)\mathbf{i} + t^2\mathbf{j}.$$

   Find the tangential and normal components of the acceleration at t.

   Answer:

   The velocity is $\mathbf{v} = \mathbf{r}'(t) = (1, 2t)$ and the acceleration $\mathbf{a} = \mathbf{r}''(t) = (0, 2)$. The acceleration in the basis of the unit tangential $\mathbf{T}$ and unit normal $\mathbf{N}$ vectors is

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}. \tag{1}$$

   We recall that the tangential vector is the unit vector with same direction as the velocity, that is $\mathbf{T} = \mathbf{r}'/\left\|\mathbf{r}'\right\|$. To find a_T we dot both sides of (1) by $\mathbf{T}$ and use that $\mathbf{T} \cdot \mathbf{T} = 1$ and $\mathbf{N} \cdot \mathbf{T} = 0$. Thus,

$$a_T = \mathbf{a} \cdot \mathbf{T} = (0, 2) \cdot \frac{(1, 2t)}{\sqrt{1 + 4t^2}} = \frac{4}{\sqrt{1 + 4t^2}}.$$

Now that we know a_T, we can find a_N very easily using the Pythagorean theorem:

$$a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2} = \sqrt{4 - \frac{16t^2}{1+4t^2}} = \frac{2}{\sqrt{1+4t^2}}.$$

4. Find the gradient vector of

$$F(x, y, z) = x^2 + 2y^2 + z^2 + 12$$

at $\mathbf{p}_0 = (1, 1, 1)$. Write the equation of the tangent plane to the surface $F(x, y, z) = 16$ at $\mathbf{p}_0$.

Answer:

We compute the gradient $\nabla F(\mathbf{p}_0) = (2x, 4y, 2z)$, which evaluated at $\mathbf{p}_0$ gives $\nabla F(\mathbf{p}_0) = (2, 4, 2)$. The point $\mathbf{p}_0$ belongs to the surface since $F(\mathbf{p}_0) = 16$. We known that the tangent plane to the surface $F(x, y, z) = 16$ goes through $\mathbf{p}_0$ and is orthogonal to the gradient $\nabla F(\mathbf{p}_0)$. Therefore its equation is given by $\nabla F(\mathbf{p}_0) \cdot (\mathbf{p} - \mathbf{p}_0) = 0$, which in the case of $\mathbf{p}_0 = (1, 1, 1)$ gives

$$2(x-1) + 4(y-1) + 2(z-1) = 0.$$

Note: I we had $z = f(x, y)$ instead, this is covered by what we did above. Indeed take $F(x, y, z) = 0$, where $F(x, y, z) = f(x, y, z) - z$. Then the equation of the tangent plane at (x_0, y_0, z_0) with $z_0 = f(x_0, y_0)$ is given by,

$$\begin{pmatrix} \nabla f \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = 0.$$

This translates into the familiar equation of a tangent plane (that can be recovered using Taylor's theorem):

$$z = f(x_0, y_0) + \nabla f(x_0, y_0) \cdot (x - x_0, y - y_0).$$

5. Find the minimum distance between the point $(1, 1, 0)$ and the surface $z = \sqrt{1 + x^2 + y^2}$.

Answer:

Points belonging to the surface are of the form $(x, y, \sqrt{1 + x^2 + y^2})$. The distance (squared, which is recommended because otherwise the distance function is not differentiable at the origin) from such a point to $(1, 1, 0)$ is given by

$$\begin{aligned} \left\| (x, y, \sqrt{1 + x^2 + y^2}) - (1, 1, 0) \right\|^2 &= (x-1)^2 + (y-1)^2 + 1 + x^2 + y^2 \\ &= 2x^2 + 2y^2 - 2x - 2y + 3 =: f(x, y). \end{aligned}$$

Thus we need to solve the unconstrained optimization problem

$$\min f(x,y).$$

This can be achieved by first finding the critical points, which are all stationary points for f, that is points (x,y) for which $\nabla f(x,y) = \mathbf{0}$. We get that $\nabla f(x,y) = (4x-2, 4y-2)$, and the stationary point $(1/2, 1/2)$. To check if this is a minimizer we use the second partials test. We compute,

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 4 & 0 \\ 0 & 4 \end{vmatrix} = 16 > 0.$$

This means that the critical point at hand is not a saddle point. To check whether it is a minimum or a maximum we look at the sign of f_{xx}. In our case $f_{xx}(1/2, 1/2) = 4 > 0$, therefore the point $(1/2, 1/2)$ is a minimizer of $f(x,y)$. The minimal squared distance is $f(1/2,1/2) = 1-1-1+3 = 2$.

6. Assume we can measure the height and radius of a cylinder to within 1% accuracy. Estimate using differentials the maximum percentage error of the calculated volume.

   Answer:

   The volume of such a cylinder is $V(r,y) = \pi r^2 h$. The change in volume $\Delta V = V(r+\Delta r, h+\Delta h) - V(r,h)$ due to the measurement errors $\Delta r = \pm 0.01 r$ and $\Delta h = \pm 0.01 h$ can be estimated using differentials as follows,

$$\begin{aligned} |\Delta V| \approx |dV| &= \left| \frac{\partial V}{\partial r}\Delta r + \frac{\partial V}{\partial h}\Delta h \right| \\ &= \left| 2\pi r h(\pm r0.01) + \pi r^2(\pm h0.01) \right| \\ &\leq 2\pi r^2 h \times 0.01 + \pi r^2 h \times 0.01 = 0.03\pi r^2 h, \end{aligned}$$

   where the inequality in the chain is the triangle inequality. Thus we get that the relative error in the volume of the cylinder $|\Delta V|/V$ is approximatively 3%.

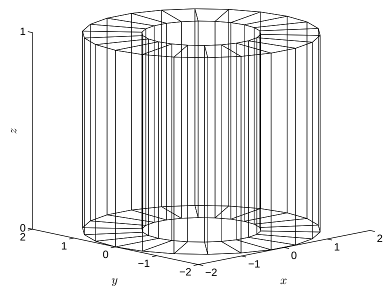
7. Evaluate the mass of a ring R of height 1, inner radius 1 and outer radius 2, where the mass density is proportional to the distance from the axis of symmetry.

   Answer:

   As can be seen from the quick sketch in figure 2, this integration begs for using cylindrical coordinates. The distance from the axis of symmetry is r, thus we take $m(r,\theta,z) = kr$, where $k > 0$ is some constant. We compute the mass as follows,

$$\begin{aligned} M &= \iiint_R m(x,y,z)dxdydz = \int_1^2 drkr^2 \int_0^1 dz \int_0^{2\pi} d\theta \\ &= k\left(\frac{2^3}{3} - \frac{1^3}{3}\right)(1)(2\pi) = \frac{14k\pi}{3}. \end{aligned}$$

Figure 2 The ring of problem 7.



8. Calculate the area $A(S)$ of the region S delimited by $y = x^2$ and $y = x$ in the following ways:

   (a) directly using double integrals

   (b) using the identity derived from Green's theorem,

   $$A(S) = \oint_{\partial S} x dy,$$

   where S has been oriented counter-clockwise as usual.

   Answer:

   See figure 3 for the region of interest and for how we partitioned the boundary ∂S of the region into two smooth paths C_1 and C_2.

   (a) Using double integrals we get that

   $$A(S) = \int_0^1 dx \int_{x^2}^{x} dy = \int_0^1 dx(x - x^2) = \frac{1}{2} - \frac{1}{2} = \frac{1}{6}.$$

   (b) Using the identity involving line integrals we get that,

   $$\begin{aligned}\oint_{\partial S} xdy &= \int_{C_1} xdy + \int_{C_2} xdy \\ &= \int_0^1 \sqrt{y}dy + \int_1^0 ydy = \frac{2}{3}y^{3/2}\Big|_0^1 + \frac{y^2}{2}\Big|_1^0 \\ &= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.\end{aligned}$$

Figure 3 The region S of interest in problem 8. The boundary is oriented counter-clockwise, and divided into C_1 ($y = x^2$, between $x = 0$ and $x = 1$) and C_2 ($y = x$, between $x = 1$ and $x = 0$).