The order of integration in triple integrals

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There are many ways of expressing a triple integral as an iterated integral, all of them should give the same result, but some are easier to evaluate by hand than others. So it pays to try different orders of integration before embarking on a long calculation. This preliminary exploratory work is done below, but it is a good exercise to rewrite the iterated integrals by yourselves.

The following integration example comes from exercise 13.7.21 in the Varberg, Purcell and Rigdon textbook. We have to determine the volume of the solid S in the first octant that is delimited by $y = 2x^2$ and the plane $y = 8 - 4z$ (see Figure 1). The volume is given by

$$\text{Vol}(S) = \iiint_S 1 dV$$

If we integrate over z first, i.e. we sum the areas of slices of the body at $z =$ constant, we get:

$$\text{Vol}(S) = \int_0^2 \int_0^{8-4z} \int_0^{\sqrt{y/2}} 1 dx dy dz \quad \text{or} \quad \text{Vol}(S) = \int_0^2 \int_0^{\sqrt{4-2z}} \int_{2x^2}^{8-4z} 1 dy dx dz,$$

depending on the order of the inner integration. Because of the square roots in the integration bounds, evaluating these integrals seems error prone. So let us try to integrate over y first. Unfortunately we also get square roots in the integration bounds,

$$\text{Vol}(S) = \int_0^8 \int_0^{\sqrt{y/2}} \int_0^{2-y/4} 1 dz dx dy \quad \text{or} \quad \text{Vol}(S) = \int_0^8 \int_0^{2-y/4} \int_0^{\sqrt{y/2}} 1 dx dz dy.$$

If the integration is done over x first, the expressions we get should be much easier to evaluate (we only have polynomials to integrate).

$$\text{Vol}(S) = \int_0^2 \int_{2x^2}^8 \int_0^{2-y/4} 1 dz dy dx \quad \text{or} \quad \text{Vol}(S) = \int_0^2 \int_0^{2-x^2/2} \int_{2x^2}^{8-4z} 1 dy dz dx$$

You can verify that $\text{Vol}(S) = 128/15$.

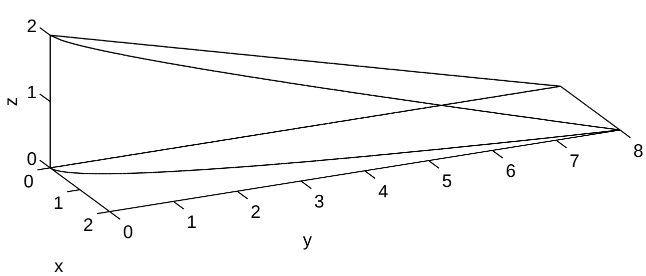

Figure 1: Solid from exercise 13.7.21.