**4.2.12:** $\alpha : [a, b] \to [\alpha(a), \alpha(b)]$ is $C^1$ and strictly monotone increasing therefore it is invertible, call $\beta(s) = \alpha^{-1}(s)$. Then

$$\beta'(s) = \frac{1}{\alpha'(\beta(s))}$$

(differentiate $\alpha(\beta(s)) = s$ using the chain rule and rearrange). The problem statement says that given an $s \in [\alpha(a), \alpha(b)]$ there is a unique $t \in [a, b]$ such that $\alpha(t) = s$. This $t$ is exactly $\beta(s)$. Thus the definition of $\mathbf{d}(s) = \mathbf{c}(t)$ is, more precisely,

$$\mathbf{d}(s) = \mathbf{c}(\beta(s)).$$

(a) Let $\mathbf{x}$ be in the image $\mathbf{c}([a, b])$ then there exists $t \in [a, b]$ such that $\mathbf{c}(t) = \mathbf{x}$, call $s = \alpha(t)$ then

$$\mathbf{d}(s) = \mathbf{c}(\beta(\alpha(t))) = \mathbf{c}(t) = \mathbf{x},$$

so $\mathbf{x}$ is in the image of $\mathbf{d}$. The other direction is similar (easier even).

(b) The arc-length is

$$\begin{aligned}
\int_{\alpha(a)}^{\alpha(b)} \|\mathbf{d}'(s)\| \, ds &= \int_{\alpha(a)}^{\alpha(b)} \|\mathbf{c}(\beta(s))'\| \, ds \\
&= \int_{\alpha(a)}^{\alpha(b)} \|\mathbf{c}'(\beta(s))\beta'(s)\| \, ds \\
&= \int_{\alpha(a)}^{\alpha(b)} \frac{1}{\alpha'(\beta(s))} \|\mathbf{c}'(\beta(s))\| \, ds \\
&= \int_a^b \frac{1}{\alpha'(t)} \|\mathbf{c}'(t)\| \, \alpha'(t) dt = \int_a^b \|\mathbf{c}'(t)\| \, dt
\end{aligned}$$

We used that $\alpha' > 0$ in the third equality, and in the fourth equality we made the $u$-substitution $t = \beta(s)$ and so

$$dt = \beta'(s) ds = \frac{1}{\alpha'(t)} dt.$$

(c) Just use chain rule

$$\frac{d}{ds}\mathbf{d}(s) = \mathbf{c}'(\beta(s))\beta'(s) = \mathbf{c}'(\beta(s))\frac{1}{\alpha'(\beta(s))} = \frac{\mathbf{c}'(\beta(s))}{\|\mathbf{c}'(\beta(s))\|}$$

which of course has norm 1. The last equality is using that $\alpha'(t) = \|\mathbf{c}'(t)\|$ by the fundamental theorem of calculus.