Stone's Theorem

A family $\mathcal{A}$ of real (or complex) valued functions on a set E is said to be an <u>algebra</u>

if (i) $f+g \in \mathcal{A}$

(ii) $f \cdot g \in \mathcal{A}$ for all $f, g \in \mathcal{A}$

(iii) $cf \in \mathcal{A}$ $c \in \mathbb{R}$ or $\mathbb{C}$

If $\mathcal{A}$ is closed under $\|\cdot\|_{sup}$ convergence then $\mathcal{A}$ is said to be <u>uniformly closed</u>

We call $\mathcal{B}$ to be the closure of $\mathcal{A}$ under uniform convergence.

e.g. $\mathbb{P}(\mathbb{R}) = \{$ polynomials w/ real coefficients $\}$ is an algebra.

or $\mathbb{P}(\mathbb{C})$

We saw that the uniform closure of $\mathcal{P}(\mathbb{R})$ in $C([a,b])$ was $C([a,b])$ i.e. $\mathcal{P}(\mathbb{R})$ is dense in $C([a,b])$ (w/ sup norm)

Thm If $\mathcal{B}$ is the uniform closure of an algebra $\mathcal{A}$ then ~~$\mathcal{B}$~~ $\mathcal{B}$ is a uniformly closed algebra.

proof $f, g \in \mathcal{B} \Rightarrow \exists f_n, g_n \in \mathcal{A}$

$$f_n \to f, \quad g_n \to g \quad \text{unif}$$

$$f_n + g_n \in \mathcal{A}, \quad f_n \cdot g_n \in \mathcal{A} \quad c f_n \in \mathcal{A}$$

$$\Rightarrow \quad f+g \in \mathcal{B}, \quad f \cdot g \in \mathcal{B}, \quad cf \in \mathcal{B}.$$

Def Let $\mathcal{A}$ be a family of fns on a set E, $\mathcal{A}$ is said to separate points of E if $x \neq y \in E$ $\exists$ ~~f, g~~ $f \in \mathcal{A}$ w/ $f(x_1) \neq f(x_2)$

If for all $x \in E$ $\exists$ $f \in \mathcal{A}$ w/ $f(x) \neq 0$ then we say $\mathcal{A}$ does not vanish on E

e.g. ~~Since to~~

~~So~~ failure to satisfy these properties is an obstruction to $\mathcal{A}$ being dense in e.g. space of cts functions

for example

$\mathcal{A} = \{$ polynomials ~~on~~ $\mathbb{R}$ w/ $p(0)=0\}$ r.v.

this is an algebra ~~since~~ but it cant be unif dense in $C([0,1])$ since all $f \in \mathcal{B}$ have $f(0)=0$.

for example

$\mathcal{A} = \{$ constant functions on $\mathbb{R}\}$

an algebra which doesnt separate points.

Thm If A an algebra on E, A separates pts and does not vanish then for any $x_1 \neq x_2 \in E$ and $c_1, c_2 \in \mathbb{R}$ $\exists$ $f \in A$ w/

$$f(x_1) = c_1 \quad \text{and} \quad f(x_2) = c_2$$

proof: $\exists$ $g, h, k \in A$ w/

$$\cancel{g(x_1) \neq f(x_2) \qquad g(x_1) \neq 0 \quad \text{and}}$$

$$g(x_1) \neq g(x_2) \quad , \quad h(x_1) \neq 0 \qquad k(x_2) \neq 0$$

x_2

x_1

now we make a basis for $\cancel{\{x_1, x_2\}}$ $\mathbb{R}^{\{x_1, x_2\}}$

$$U(x) = \cancel{g(x)k(x) - g(x_2)} \; (g(x) - g(x_1))\, k(x)$$

$$V(x) = (g(x) - g(x_2))\, h(x)$$

then $u(x_1) = 0$, $u(x_2) \neq 0$ so

$v(x_1) \neq 0$, $v(x_2) = 0$

$$f(x) = c_1 \frac{v(x)}{v(x_1)} + c_2 \frac{u(x)}{u(x_1)} \qquad \text{satisfies property}$$

<u>Thm</u> let $\mathcal{A}$ be an algebra of cts r.v. functions on a compact metric space K. ~~Then~~ If $\mathcal{A}$ separates pts and does not vanish on K then $\bar{\mathcal{A}} = C(K)$.

<u>proof</u>: 1. $f \in \mathcal{B}$ then $|f| \in \mathcal{B}$.

let $a = \sup |f|$ and $\varepsilon > 0$ there ~~is~~ are $c_1, \dots, c_n \in \mathbb{R}$ s.t.

$$\left| \sum_{i=1}^{n} c_i y^i - |y| \right| < \varepsilon \qquad (-a \le y \le a)$$

by Weierstrass thm

then $g = \sum_{i=1}^{n} c_i f^i \in \mathcal{B}$ since $\mathcal{B}$ is an algebra and since $|f| \le a$

$$\| g - \cancel{\sum_{i=1}^{n} c_i g^i}\, |f| \|_{sup} < \varepsilon$$

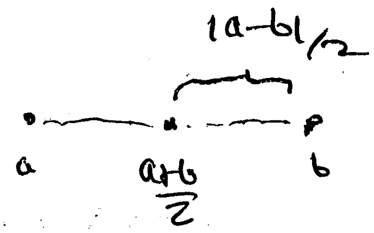
since $\mathcal{B}$ is closed $\Rightarrow |f| \in \mathcal{B}$.

2. If $f, g \in \mathcal{B}$ then $\max(f,g)$, $\min(f,g) \in \mathcal{B}$

$$\max(f,g) = \begin{cases} f(x) & \text{if } f(x) \geq g(x) \\ g(x) & \text{if } f(x) < g(x) \end{cases}$$

$$\max(f,g) = \frac{f+g}{2} + \frac{|f-g|}{2}$$

$$\min(f,g) = \frac{f+g}{2} - \frac{|f-g|}{2}$$



similarly $\max(f_1, \ldots, f_n)$ and $\min(f_1, \ldots, f_n)$ are in $\mathcal{B}$ when $f_i \in \mathcal{B}$ $i = 1, \ldots, n$

3. given $f \in C(K)$ and $x \in K$, $\varepsilon > 0$ $\exists$ $g_x \in \mathcal{B}$ w/ $g_x(x) = f(x)$ and $g_x(t) \geq f(t) - \varepsilon$ $\forall t \in K$

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since $\mathcal{A}$ sep. pts and does not vanish so does $\mathcal{B} = \bar{\mathcal{A}}$

for all $y \in K$ $\exists$ $h_y \in B$ w/

$h_y(x) = f(x)$, $h_y(y) = f(y)$

Since h_y cts $\exists$ B_y open containing y

~~and~~ s.t. $\forall$ $t \in B_y$

$h_y(t) > f(t) - \varepsilon$,

Now $K \subset \bigcup_{y \in K} B_y$ cpct $\Rightarrow$

$K \subset B_{y_1} \cup \cdots \cup B_{y_n}$

Call ~~$g_x(t) =$~~ $g_x = \max(h_{y_1}, \ldots, h_{y_n})$

for all $t \in K$, $t \in B_{y_j}$ for some j

$g_x(t) \geq h_{y_j}(t) > f(t) - \varepsilon$.

Since all of the $h_{y_j}(x) = f(x) \Rightarrow g_x(x) = f(x)$.

4. Given $f \in C(K)$, $\varepsilon > 0$ $\exists$ $h \in \mathcal{B}$ w/

$\| h - g \|_{sup} < \varepsilon$

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almost the same as step 3

Let $x \in K$ and g_x as above

by continuity $\exists$ V_x open s.t.

$g_x(t) < f(t) + \varepsilon$ for $t \in V_x$

$K \subset V_{x_1} \cup \dots \cup V_{x_m}$ ~~so take~~

for some $x_1, \dots, x_m$

take ~~h(t)~~ $h = \min(g_{x_1}, \dots, g_{x_m})$

$h \in \mathcal{B}$ and $\forall$ $t \in K$

$h(t) > f(t) - \varepsilon$ since this was true of

all the g_{x_j}

Also now $t \in K$ $\exists\, j$ s.t. $t \in V_{x_j}$ so

$$h(t) \leq g_{x_j}(t) < f(t) + \varepsilon$$

$$\Rightarrow \quad \| h - f \|_{sup} < \varepsilon \quad \square$$

For complex algebras need to actually have $*$-algebra

(iv) $f \in A \Rightarrow f^* \in A$

complex conjugate.

Then can use result for real algebras.