# Functions of Several Variables

first let us recall some linear Algebra (which you should have taken last quarter, if you have not for some reason read beginning of Rudin Ch. 9 for a very brief primer)

- Vector Space $V$ comes w/ addition and scalar multiplication (scalars in $\mathbb{R}$, $\mathbb{C}$ or some other field)
- for $x_1, \dots, x_k \in V$ $c_1, \dots, c_k \in \mathbb{R}$

$$c_1 x_1 + \dots + c_k x_k$$ called a linear combination

- If $S \subset V$ ~~and E~~ we define

$$\text{span } S = \{\text{linear combinations of elements of } S\}$$

- A set of vectors $x_1, \dots, x_k$ is called independent if

$$c_1 x_1 + \dots + c_k x_k = 0 \text{ iff } c_1 = c_2 = \dots = c_k = 0.$$

~~A set vector space $V$ has dimension $d$ if~~

- An independent set of $V$ which spans $V$ is called a <u>basis</u> for $V$.

e.g. $\{e_1, \dots, e_k\}$ is a basis for $\mathbb{R}^k$

if $\{x_1, \dots, x_k\}$ is a basis for $V$ then every element $x \in V$ can be written in a <u>unique</u> way as

$$x = \sum_{j=1}^{k} c_j x_j .$$

$c_1, \dots, c_k$ are the <u>coordinates</u> of $x$ wrt the basis $\{x_1, \dots, x_k\}$.

- A vector space has dimension $k$ if it has a basis w/ $k$ vectors.

- Standard basis on $\mathbb{R}^n$
- Normed Vector Spaces
- Inner product spaces

A <u>Linear transformation</u> is a mapping

$$A: X \to Y$$

for $X, Y$ vector spaces such that

$$A(x_1 + x_2) = Ax_1 + Ax_2$$

i.e. $A$ is <u>linear</u>

$$A(cx) = cAx$$

note that if $A$ is linear then

$$A0 = 0.$$

We call

$$L(X,Y) := \{A : X \to Y : A \text{ linear}\}$$

and $L(X) = L(X,X)$ for simplicity

if we choose to specify bases

$$x_1, \dots, x_n \quad \text{and} \quad y_1, \dots, y_m$$

for $X$ and $Y$ resp

then $A$ can be represented as a matrix

$$Ax = \sum_{j=1}^{n} c_j A x_j$$ so $A$ is determined

by its action on the basis vectors

define $a_{ij}$ s.t.

$$A x_j = \sum_{i=1}^{m} a_{ij} y_i \qquad \text{for } 1 \le j \le n.$$

Then for $x \in X$ w/ $x = \sum_{j=1}^{n} c_j x_j$

$$Ax = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} a_{ij} c_j \right) y_i$$

~~they~~

We often use the matrix notation to compute this sort of thing

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix}$$

Then the columns of $A$ are the coordinate representations of $Ax_j$

~~$[A][x] =$~~

$$Ax = \sum_{j=1}^{n} c_j \begin{bmatrix} a_{1j} \\ \vdots \\ \vdots \\ a_{mj} \end{bmatrix} = \sum_{j=1}^{n} c_j Ax_j$$

linear combination of the column vectors

i.e. range $A$ = span(column vectors of $A$)

Similarly given fixed bases for $X, Y$

and an $m \times n$ matrix of real entries we identify a unique element of $L(X, Y)$.

<u>Theorem</u> A linear operator $A$ on a finite dimensional vector space is one-to-one if and only if it is onto.

In terms of the matrix representation

$A$ is 1-1 if and only if

$0 = Ax = \sum_j c_j Ax_j$ iff $c_j = 0 \quad \forall \; 1 \le j \le n$

i.e. iff the column vectors of $[A]$ are linearly independent.

~~$A$ is onto~~

If $X, Y, Z$ vector spaces

$A \in L(X, Y)$ , $B \in L(Y, Z)$

then $AB x$ ~~$BA = B(Ax)$~~

$BAx = B(Ax)$ composition

$BA \in L(X, Z)$

We define a <u>norm</u> of linear operators $A \in L(X,Y)$

$$\|A\| = \sup_{|x| \le 1} |Ax|$$ Called <u>operator norm</u>

From the definition for all $x \in X$

$\cancel{\|Ax\| \le \|A\| |x|}$

$$\boxed{|Ax| \le \|A\| \, |x|}$$

(divide by $|x|$ on both sides as long as $|x| \neq 0$)

<u>Thm</u> (a) ~~if $\|A\| < \infty$ then A is~~
for every $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, $\|A\| < \infty$ and
A is a uniformly continuous mapping $\mathbb{R}^n$ to $\mathbb{R}^m$.

(b) $A, B \in L(X,Y)$ c a scalar

$\|A+B\| \le \|A\| + \|B\|$ and $\|cA\| = |c| \, \|A\|$

(i.e. $\|\cdot\|$ is indeed a norm)

(c) If $A \in L(X,Y)$, $B \in L(Y,Z)$ then

$\|AB\| \le \|A\| \|B\|$.

proof: (a) for all $|x| \le 1$.

$$|Ax| = \left|\sum_{i=1}^{n} c_i Ae_i\right|$$

$$\le \sum_{i=1}^{n} |c_i|\,|Ae_i|$$

$$\le \sum_{i=1}^{n} |Ae_i| < +\infty$$

$x = \sum_i c_i e_i$

$1 \ge |x| \ge |c_i|$ for each $i$,

$$|Ax - Ay| = |A(x-y)| \le \|A\|\,|x-y|$$

so ~~Ax~~ $A$ is in fact Lipschitz continuous.

(b) $$|(A+B)x| = |Ax + Bx| \le |Ax| + |Bx|$$

$$\le (\|A\| + \|B\|)\,|x|$$

$$\Rightarrow \quad \|A+B\| \le \|A\| + \|B\|$$

similarly for (c)

$$|BAx| \le \|B\|\,|Ax| \le \|B\|\,\|A\|\,|x|,$$

So $L(X)$, $L(X,Y)$ are normed vector spaces themselves, one can easily check that they are complete as well.

Thm $L(X)$, $L(X,Y)$ are Banach spaces

Thm Let $\Omega = \{$ invertible linear operators on $\mathbb{R}^n \}$

(a) if $A \in \Omega$ and $B \in L(\mathbb{R}^n)$ w/

$$\|B-A\|\|A^{-1}\| < 1$$

then $B \in \Omega$.

(b) $\Omega$ is an open subset of $L(\mathbb{R}^n)$ and the mapping $A \mapsto A^{-1}$ is continuous.

33

proof: We aim to show $B$ is 1-1

for linear operators this is essentially the same as showing $|Bx| \geq c|x|$ for some $c > 0$

since this translates as an upper bound for $\|B^{-1}\| \leq \frac{1}{c}$.

~~$|Bx| = |(B-A)x + Ax| \geq |Ax| - |(B-A)x|$~~

~~$\geq |Ax| - \|B-A\| |x|$~~

Indeed for $A$ (which is invertible)

~~Ax~~ $|x| = |A^{-1}Ax| \leq \|A^{-1}\| |Ax|$

Then we use this in

$$|Bx| = |(B-A)x + Ax| \geq |Ax| - |(B-A)x|$$

$$\geq \|A^{-1}\|^{-1} |x| - \|B-A\| |x|$$

$$= \underbrace{\|A^{-1}\|^{-1}(1 - \|A^{-1}\| \|B-A\|)}_{>0} |x|$$

so $Bx = 0$ iff $x = 0$ i.e. $B$ is 1-1

Also we can now get, setting $x = B^{-1}y$

above

$$|B^{-1}y| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\| \|B-A\|} |y| ,$$

i.e. $$\|B^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\| \|B-A\|}$$

That $\Omega$ is open is clear from part (a)

for the continuity use the identity

$$B^{-1} - A^{-1} = B^{-1}(A-B)A^{-1}$$

so

$$\|B^{-1} - A^{-1}\| \le \|B^{-1}\| \|A^{-1}\| \|A-B\|$$

$$\le \frac{\|A^{-1}\|}{1 - \|A^{-1}\| \|A-B\|} \|A-B\|$$

~~so $B^{-1}$~~ $\Rightarrow$ continuity of the map $A \mapsto A^{-1}$ at $A$.

$\square$

in terms of the matrix entries w.r.t. standard basis

$$|Ax|^2 = \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} a_{ij} c_j\Big)^2 \le \sum_{i=1}^{m} \Big(\sum_j a_{ij}^2\Big)\Big(\sum_j c_j^2\Big)$$

$$= \Big(\sum_{i,j} a_{ij}^2\Big) |x|^2$$

Thus $\|A\| \le \Big(\sum_{i,j} a_{ij}^2\Big)^{1/2}$

i.e. if $S$ metric space

$S \ni p \to \{a_{ij}\} \in \mathbb{R}^{m \times n}$ is cts map

then $p \mapsto A_p$ where $A_p \in L(\mathbb{R}^n, \mathbb{R}^m)$

has coefficients $a_{ij}$ wrt orthonormal bases on $\mathbb{R}^n$, $\mathbb{R}^m$

~~then $A_p$ is cts wrt. $p$.~~

then $p \mapsto A_p$ cts map from $S \to L(\mathbb{R}^n, \mathbb{R}^m)$