... and Series of functions

We will mostly talk about functions

$$f : \cancel{X}^{E} \to \mathbb{R}$$

↑ set ~~metric space~~

but the results can be extended to

vector valued f or even metric space valued

without any serious difficulty.

~~unless~~ (what space is appropriate for continuity / differentiation / integration)

suppose $f_n : E \to \mathbb{R}$

↑ a set (no structure)

and $f_n(x) \in \mathbb{R}$ converges for every $x \in E$

then we can define f

$$f(x) = \lim_{n\to\infty} f_n(x) \quad (x \in E)$$

The pointwise limit or limit function

and we say f_n converges to f pointwise on E

Similarly one can define

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

if f is the pointwise limit of the partial sums $F_N(x) = \sum_{n=1}^{N} f_n(x)$

- What kind of properties are ~~exchanged~~ preserved under limits?

1. continuity?
2. differentiability? do derivatives converge too?
3. Integrability? do integrals converge too?

All of these questions have to do with

exchanging limits

(as I said one of the fundamental topics of this course

5 asking if f_n cts $\Rightarrow$ f is cts is asking if

$$\lim_{t\to x} f(t) = \lim_{t\to x}\lim_{n\to\infty} f_n(t) \overset{(?)}{=} \lim_{n\to\infty}\lim_{t\to x} f_n(x)$$

$$= \lim_{n\to\infty} f(x)$$

such exchange is <u>not</u> always allowed

typically need something more than ptwise limits

<u>Example</u>: $s_{m,n} = \frac{m}{m+n}$ i.e. a sequence (in n) of fcns on $\mathbb{N}$.

$$\lim_{n\to\infty} s_{m,n} = 0 \quad , \quad \lim_{m\to\infty} s_{m,n} = 1$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\lim_{m\to\infty}\lim_{n\to\infty} s_{m,n} = 0 \qquad \lim_{n\to\infty}\lim_{m\to\infty} s_{m,n} = 1$$

example: $f_n(x) = \cancel{(1-|x|)} \ (1-n|x|)_+ = \begin{cases} 1-n|x| & \text{for } |x| \le \frac{1}{n} \\ 0 & \text{for } |x| \ge \frac{1}{n} \end{cases}$

$f_n(x) \to 0$ as $n \to \infty$ for $x \neq 0$

$f_n(0) = 1$ for all n if $x = 0$

so $f_n \to f(x) = \begin{cases} 1 & x=0 \\ 0 & \text{else} \end{cases}$ discontinuous

even though all the f_n were ct.

example: $f_n(x) = \begin{cases} 1 & x \ge \frac{1}{n} \\ -n & 0 < x < \frac{1}{n} \\ 1 & x = 0 \end{cases}$ $f_n \in \mathcal{R}$ on $[0,1]$

$$\int_0^1 f_n(x)\,dx = 1\cdot\left(1-\frac{1}{n}\right) - n\cdot\frac{1}{n} = -\frac{1}{n} \to 0 \quad \text{as } n\to\infty$$

but $f_n(x) \to f(x) = 1$ as $n \to \infty$

$\int_0^1 f(x)\,dx = 1 \neq 0$.

Convergence

Pointwise convergence is not a strong enough concept to preserve properties like

- Integration
- continuity
- differentiation

which are also based of limiting procedures.

We say $f_n \to f$ <u>uniformly</u> on E if

for all $\varepsilon > 0$ $\exists$ N s.t. $n \geq N$

$$\Rightarrow \quad |f_n(x) - f(x)| < \varepsilon \quad \underline{\text{for all}} \ x \in E.$$

- if $f_n \to f$ unif it does pointwise as well.

for unif convergence given an ~~ε~~ ε there is <u>one</u> N that works <u>for</u> all x.

Since the target space $\mathbb{R}$ is complete

Thm Sequence $f_n : E \to \mathbb{R}$ converges uniformly iff

$\forall \ \varepsilon > 0 \quad \exists \ N$ s.t. $n, m \geq N \Rightarrow$

$$|f_n(x) - f_m(x)| < \varepsilon \quad \forall \ x \in E.$$

proof: exercise $\square$

~~Conv~~ Uniform convergence is the same as converges in supremum norm

$$V = \{ f : E \to \mathbb{R} \} \quad \text{vector space}$$

$$\| f \|_{sup} = \sup_E f$$

Thm $f_n \to f$ unif on E iff

$$\| f_n - f \|_{sup} \to 0 \quad \text{as} \quad n \to \infty.$$

M criterion for unif convergence of sums.

just need tail to be unif small

Thm Spse $f_n : E \to \mathbb{R}$ and $|f_n| \le M_n$ on E and

$\sum M_n < +\infty$ then $\sum_{n=1}^{\infty} f_n$ converges uniformly

Proof: Cauchy criterion: let N s.t. $\sum_{n \ge N} M_n < \varepsilon$

then $\forall\ n, m > N$

~~$|f_{n+1} - f_m$~~

$$\left| \sum_{j=1}^{n} f_j - \sum_{j=1}^{m} f_j \right| \le \sum_{n}^{m} M_j < \varepsilon$$

$$\Rightarrow \quad \sum_{j=1}^{n} f_j \xrightarrow[\text{unif}]{} \text{ to some } f : E \to \mathbb{R}.$$

Continuity

Thm Spse $f_n \to f$ unif on E ~~subset in~~ metric space. ~~then~~

~~let $x \in E$~~ and

let x a limit point of E and $\lim_{t \to x} f_n(t) = A_n$

then A_n converge and $\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n$

Thm Let X metric space and $f_n : X \to \mathbb{R}$ cts.

If $f_n \to f$ uniformly then f is cts.

Cor) The ~~metric~~ normed space $C(X) = \{f : X \to \mathbb{R} : f \text{ cts}\}$

w/ $\|f\|_{C(X)} = \sup_X |f|$ is complete.

proof: Let $x \in X$ and $\varepsilon > 0$, there is

N s.t. $n \geq N \Rightarrow \|f_n - f\|_\infty < \varepsilon/3$

Let $\delta > 0$ s.t. $|y - x| < \delta \Rightarrow |f_N(x) - f_N(y)| < \varepsilon/3$

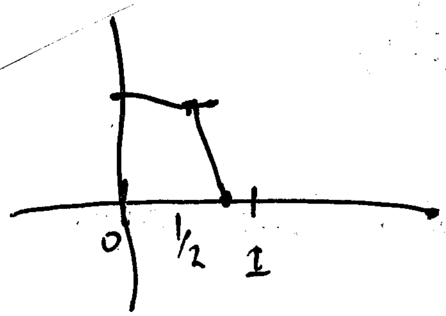
then $|f(x) - f(y)| \leq |f_N(x) - f(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(x)|$

$\leq 2\|f_N - f\|_\infty + \varepsilon/3$

$< \varepsilon$

$\Rightarrow f$ is cts at x.

$f_n(x)$ cts, $f_n \to f$ ptwise, f disconts



$X = [0,1]$

$$f_n(x) = \begin{cases} 1 & x \le \frac{1}{2} \\ 1 - n(x - \frac{1}{2}) & \frac{1}{2} \le x \le \frac{1}{2} + \frac{1}{n} \\ 0 & \frac{1}{2} + \frac{1}{n} < x \le 1 \end{cases}$$

$$f_n \to f(x) = \begin{cases} 1 & x \le \frac{1}{2} \\ 0 & x > \frac{1}{2} \end{cases} \quad \text{ptwise}$$

but not unif, and f is discts

~~Later we will comp come~~

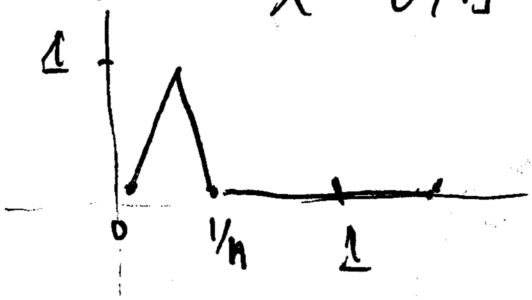
You may wonder if $f_n \to f$ ptwise and f is cts does this mean that the convergence is uniform? (kind of converse to Thm)

Answer: No in general e.g.

$(1 - nx)_+$ on $X = (0,1)$ (missing compactness)

$\longrightarrow$

on $X = [0,1]$ take



$$f_n(x) = (1 - n|x - \tfrac{1}{n}|)_+$$

$f_n \to 0$ ptwise not unif.

<u>Thm</u>: Suppose K cpct and (i) f_n cts on K

(ii) $f_n \to f$ ptwise on K, f cts

(iii) $f_{n+1} \le f_n$ on K $\forall$ ~~$1 \le n$~~ $n = 1, 2, \ldots$

Then $f_n \to f$ uniformly.

<u>proof</u>: Call $g_n = f_n - f$, g_n cts, $g_n \xrightarrow[n\to\infty]{} 0$

and $g_{n+1} \le g_n$.

Call $K_n = \{g_n \geq \varepsilon\}$, K_n are nested $K_{n+1} \subseteq K_n$

since $g_{n+1} \geq \varepsilon \Rightarrow g_n \geq \varepsilon$

K_n cpct since g cts and K cpct (closed subset of cpct is cpct) each $x \in K$ is in only finitely many K_n

$$\Rightarrow \bigcap_{n=1}^{\infty} K_n = \emptyset$$

$$\Rightarrow \quad K_N = \emptyset \quad \text{for some } N$$

(by nested intersection property)

$$\Rightarrow \quad g_n < \varepsilon \quad \text{for} \quad n \geq N. \qquad \square$$

Integration

Thm Suppose $f_n \in R$ on $[a,b]$ for $n = 1, 2, \ldots$ and $f_n \to f$ unif on $[a,b]$. Then $f \in R$ on $[a,b]$ and

$$\int_a^b f \, dt = \lim_{n\to\infty} \int_a^b f_n \, dt$$

pf:

$$\int_a^b f_n - \|f_n - f\|_\infty \, dt \le \underline{\int_a^b} f \, dt \le \overline{\int_a^b} f \, dt \le \int_a^b f_n + \|f_n - f\|_\infty \, dt$$

since

$$f_n - \|f_n - f\|_\infty \le f \le f_n + \|f_n - f\|_\infty$$

so letting $n \to \infty$ $\|f_n - f\|_\infty \to 0$

$\Rightarrow$ $f \in \mathcal{R}$ on $[a,b]$ and

$$\left| \int_a^b f \, dt - \int_a^b f_n \, dt \right| \le (b-a) \|f_n - f\|_\infty$$

(which is easy from Δ-ineq if you know $f \in \mathcal{R}$)

$\square$

Cor If $f_n \in \mathcal{R}$ on $[a,b]$ and $f(x) = \sum_{n=1}^\infty f_n(x)$ w/ the series conv. unif then

$$\int_a^b f \, dt = \sum_{n=1}^\infty \int_a^b f_n \, dt .$$

Differentiation

$f_n \to f$ unif not sufficient for concluding

then f diff or $f_n' \to f'$

Thm: Suppose $f_n : [a,b] \to \mathbb{R}$ diff, $f_n(x_0)$ converges

and f_n' are $\boxed{\text{cts}}$ and converge uniformly on $[a,b]$.

Then f_n converge uniformly on $[a,b]$ to a fcn f

and $$f'(x) = \lim_{n\to\infty} f_n'(x)$$

proof: Write $$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(t)\,dt$$

using fundamental thm of calc (f_n' cts)

f_n' converge uniformly to a function g

$\Rightarrow$ g is continuous and Riemann Integrable

Call $$f(x) = \lim_{n\to\infty} f_n(x_0) + \int_{x_0}^{x} g'(t)\,dt$$

Then

$$|f(x) - f_n(x)| \le |f(x_0) - f_n(x_0)| + \left|\int_{x_0}^{x} g(t) - f_n'(t)\,dt\right|$$

$$\le |f(x_0) - f_n(x_0)| + \|g - f_n'\|_{sup}\,|x - x_0|$$

$$\le |f(x_0) - f_n(x_0)| + \|g - f_n'\|_{sup}\,(b-a)$$

$f_n' \to g$ unif and $f_n(x_0) \to f(x_0)$

$\Rightarrow$ $f_n \to f$ uniformly

but again by FTC

$$f'(x) = g(x) = \lim_{n\to\infty} f_n'(x)$$

$\square$