~~Comp~~

Sequential compactness for ~~continuous~~ functions

Given a sequence $f_n : E \to \mathbb{R}$ what kind of conditions can we put to get subsequential convergence:

~~For~~ We certainly need $f_n(x)$ to be bounded for each $x \in E$

(for sequences of real numbers have convergent subsequence (more or less) need boundedness)

Def We say f_n <u>uniformly bounded</u> on E if $\exists\ M > 0$ s.t.

$$|f_n(x)| \leq M \qquad \forall\, x \in E,\ \forall\, n \in \mathbb{N}.$$

~~A slightly~~

if f_n uniformly bdd and $E_1 \subset E$ is countable then we can find f_{n_k} converging ptwise on E_1.

Subsequence Diagonal argument:

$E_1 = \{x_1, x_2, \ldots\}$

let subseq $f_{n_{1,k}}$ s.t.

$f_{n_{1,k}}(x_1)$ converges

let $f_{n_{2,k}}$ a further subseq of $f_{n_{1,k}}$ s.t.

$f_{n_{2,k}}(x_2)$ converges

and so on.

$$\begin{array}{cccc} f_{1,1} & f_{1,2} & f_{1,3} & \cdots \\ f_{2,1} & f_{2,2} & f_{2,3} & \cdots \\ f_{3,1} & f_{3,2} & f_{3,3} & \cdots \\ \vdots & \vdots & & \end{array}$$

take the <u>diagonal subseq</u> $f_{k,k}$

then $f_{k,k}(x_j)$ converges as $k \to \infty$

for each $j \in \mathbb{N}$.

In general this is not enough to make f converge everywhere

Example $f_n(x) = \sin(nx)$

Equicontinuity

To get an everywhere convergent subseq we will need to put a stronger assumption.

(~~hard~~ impossible? to make an assumption to get just ptwise ~~convergence~~ ~~since~~ sequential cpct)

Def: A family $\mathcal{F}$ of functions $f: X \to \mathbb{R}$ (metric space X) is called equicontinuous if $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ s.t.

$$|f(x) - f(y)| < \varepsilon \text{ for all } d(x,y) < \delta$$

and all $f \in \mathcal{F}$.

Thm If K is a compact metric space and $f_n \in C(K)$ s.t. f_n converges unif on K then f_n is equicontinuous in K.

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i.e. $\mathcal{F}$ ~~equicontinuous~~ $\subseteq C(K)$

sequentially compact $\Rightarrow$ equicontinuous and unif bounded

Thm If $\mathcal{F} \subseteq C(K)$ is compact then $\mathcal{F}$ is uniformly bdd and equicontinuous.

Proof: ~~$\mathcal{F}$~~ Cpct $\Rightarrow$ bounded (in sup norm in this case)

$\Rightarrow$ uniformly bounded

$\|f\|_{sup} \leq M \quad \forall f \in \mathcal{F}$

for some $M > 0$

Let $\varepsilon > 0$, $\exists$ $f_1, \ldots, f_N$ s.t. $\forall f \in \mathcal{F}$

$\|f - f_j\|_{sup} \leq \varepsilon/3$ for ~~all~~ some $1 \leq j \leq N$

Now f_j are cts $\Rightarrow$ (K cpct) unif cts

so $\exists$ $\delta = \min_{1 \le j \le N} \delta_j$ s.t.

for $d(x,y) < \delta$, $|f_j(x) - f_j(y)| < \varepsilon/3$ for all $1 \le j \le N$.

Now let $f \in \mathcal{F}$ arbitrary, $d(x,y) < \delta$

and f_j s.t. $\|f_j - f\|_{sup} < \varepsilon/3$ then

$$|f(x) - f(y)| \le |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)|$$

$$\le 3 \cdot \varepsilon/3 = \varepsilon$$

$\Rightarrow$ $\mathcal{F}$ is equicontinuous family.

$\square$

In particular convergent sequences form cpct sets so $f_n \to f$ unif on K

$\Rightarrow$ f_n unif bdd and equicontinuous.

Thm If $\mathcal{F} \subset C(K)$ is uniformly bounded and equicontinuous then $\mathcal{F}$ is compact in $C(K)$.

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i.e. if sequence f_n is unif bdd and equicontinuous then it has a subsequence f_{n_k} converging uniformly on K.

This is called the Arzela-Ascoli Theorem

~~on~~ one of the most important results of this class,

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proof: Since K is compact, it is separable

let $E \subset K$ be a countable dense set

By previous result any sequence $f_n \in \mathcal{F}$ has a subsequence f_{n_k} which converges on E.

Lets just call this f_k and forget about originical sequence

Let $\varepsilon > 0$ and pick δ from the equicontinuity

s.t. $|f_k(x) - f_k(y)| < \varepsilon$ if $d(x,y) < \delta$

for all $k \in \mathbb{N}$.

Since E dense $K \subset \bigcup_{x_i \in E} B(x_i, \delta)$

K cpt $\Rightarrow$ $K \subset B(x_1, \delta) \cup \cdots \cup B(x_m, \delta)$

for some $x_1, \ldots, x_m \in E$

Since f_k converge on E there is

$N > 0$ s.t. $k \geq N \Rightarrow$

$|f_k(x_i) - f(x_i)| < \varepsilon \quad \forall\ 1 \leq i \leq m.$

Then for $x \in K$, $\exists\ x_i$ w/ $d(x_i, x) < \delta$

$\Rightarrow$

$$|f_k(x) - f(x)| \leq |f_k(x) - f_k(x_i)| + |f_k(x_i) - f(x_i)| + |f(x_i) - f(x)|$$

$$\leq 3\varepsilon$$

so $\|f_k - f\|_{\sup} < \varepsilon$ for $k \geq N$. $\square$