Existence for ODE initial value Problems

look for a solution X_t of the following problem

$$(1)\quad \begin{cases} \dot{X}_t = f(t, X_t) & \text{for } t>0 \qquad X_t \in \mathbb{R}^n \\ X_0 = x_0 & \qquad\qquad f: \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n \end{cases}$$

here $\dot{X}_t$ is just a notation for $\frac{d}{dt} X_t$

We will prove two versions of the existence theorem

Picard's Thm

Here we assume that $f(t,\cdot)$ is Lipschitz continuous uniformly in $t \in \mathbb{R}$.

i.e. $\exists\ L>0$ s.t. $\forall\ t\in\mathbb{R},\ X, Y \in \mathbb{R}^n$

$$(2)\quad \left| f(t,X) - f(t,Y) \right| \leq L|X-Y| \quad \text{and } f \text{ cts in } t \text{ variable}$$

Instead of solving the ODE we will solve its <u>integral form</u>

$$(3)\quad X_t = x_0 + \int_0^t f(s, X_s)\, ds$$

~~by FTC~~ If X_t solves (3) then X_t is cts

$\Rightarrow$ (by FTC and the formula (3)) that

X_t is differentiable and

$$\dot{X}_t = f(t, X_t) .$$

Thm: Suppose that f satisfies above assumption. Then there exists a global in time, cts solution to the integral equation (3).

proof: We will argue using the contraction mapping theorem.

Consider the functional $\Phi : C([0,T]) \to C([0,T])$

$$\Phi(X_\cdot) = x_0 + \int_0^t f(s, X_s)\, ds$$

defined for every cts path on $[0,T] \to \mathbb{R}^n$

since $f(t, X_t)$ is cts $\Rightarrow$ R.I.

$\Phi(X_\cdot)(t)$ is cts in t since

X_t cts on $[0,T] \Rightarrow$ bdd $\Rightarrow f(t, X_t)$ bdd by some M

$$\left|\Phi(X_\cdot)(t) - \Phi(X_\cdot)(u)\right| = \left|\int_u^t f(s, X_s)\, ds\right| \leq M|t-u|$$

how show Φ is a contraction for T small.

$$\left|\Phi(X_\cdot) - \Phi(Y_\cdot)\right| = \left|\int_0^t f(s,X_s) - f(s,Y_s)\,ds\right|$$
$$\leq \int_0^t |f(s,X_s) - f(s,Y_s)|\,ds$$
$$\leq \int_0^t L\,|X_s - Y_s|\,ds$$
$$\leq LT\,\|X-Y\|_{\sup[0,T]}$$

$$\Rightarrow \quad \|\Phi(X) - \Phi(Y)\|_{\sup[0,T]} \leq LT\,\|X-Y\|_{\sup[0,T]}$$

for $T < \frac{1}{2L}$ $\Rightarrow$ Φ is a contraction.

of $C([0,T]) \to C([0,T])$

so Φ has a unique fixed pt X^* which solves

$$X^*_t = x_0 + \int_0^t f(s,X^*_s)\,ds = \Phi(X^*_\cdot)(t)$$

how to extend X_t from solution on $[-\frac{1}{2L}, \frac{1}{2L}]$

to whole real line just note that can take

$X_{\frac{1}{2L}}$ as initial data at $t = \frac{1}{2L}$ and

find solution on $[\frac{1}{2L}, \frac{1}{4L}]$ etc. $\square$

when $f(t,X)$ is just locally Lipschitz in X

e.g if $D_X f$ is cts (i.e, bdd on cpct sets)

then one has to be more careful ~~stm~~

e.g:
$$\begin{cases} \dot{X}_t = X^2 & \text{for } t>0 \\ X_0 = x_0 > 0 \end{cases} \qquad X_t \in \mathbb{R}.$$

solution is $\cancel{\frac{x_0}{\sqrt{1-tx_0}}}$ blows up at $t = 1/x_0$

$$X_t = \frac{x_0}{1-x_0 t}$$

i.e no global in time solution.

Thm Suppose f is cts and $\frac{\partial f}{\partial X}$ is cts on open $U \subseteq \mathbb{R}^n$, then for every x_0, $\exists$ a ~~uni~~ time $T(x_0)$ s.t. there is a unique cts solution of (3) on $[-T(x_0), T(x_0)]$.

proof: We try to do the same contraction mapping argument but we need to be more careful

since $D_X f$ cts on any cpct ~~K~~ $\overline{B(x_0,r)} \subset U$ containing x_0

$\exists$ $M, L > 0$ s.t. $\|D_x f\| \leq L$ $|f| \leq M$

define the metric space for $T = \frac{r}{M} \wedge \frac{1}{2L}$

$$A = \left\{ X_t : [0,T] \to \mathbb{R}^n \;\middle|\; \begin{array}{l} X_0 = x_0,\ X_\cdot \text{ cts and} \\ \|X_t - x_0\|_{sup} \le r \end{array} \right\}$$

this is a closed subspace of $C([0,T]) \Rightarrow$ complete

then define Φ as before

$$\Phi(X_\cdot) = x + \int_0^t f(s, X_s)\, ds$$

$\Phi(X)(0) = x_0$,

$\Phi(X)$ cts when X cts

$$\left|\Phi(x) - x_0\right| \le \int_0^t |f(s, X_s)|\, ds \quad \cancel{\le Mt \le MT}$$

$$\le Mt \le MT \le r$$

so $\Phi : A \to A$

as before Φ is a contraction

$$\left|\Phi(X) - \Phi(Y)\right| \le \int_0^t |f(s, X_s) - f(s, Y_s)|\, ds$$

$$\le \int_0^t L|X_s - Y_s|\, ds \le LT\|X - Y\|_{sup[0,T]}$$

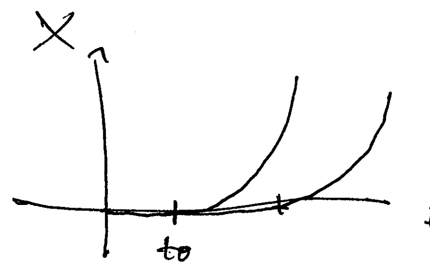
$$\le \frac{1}{2}\|X - Y\|_{sup[0,T]} \qquad \square$$

Peano Existence Thm

The other direction to extend this result is to remove the Lipschitz requirement.

The issue is that such ODE's no longer have uniqueness so fixed pt argument will not work.

example
$$\begin{cases} \dot{X}_t = X^{1/2} \\ X_0 = 0 \end{cases}$$



family of solutions $X_t = \frac{1}{4}(t - t_0)_+^2$ for $t_0 > 0$

Instead of contraction mapping we will use compactness argument, as with contraction mapping we iterate

(4) $$X^n_t = x_0 + \int_0^t f(s, X^{n-1}_s)\, ds \quad \text{w/ } X^0_t = x_0$$

but now we use eqn (4) to show that iterates are unif bdd and equicontinuous $\Rightarrow$ pre-compact in $C([0,T])$.

Well I think that sequence of iterates works, but lets do it the way Rudin suggests (problem 25 chapter 7)

<u>Theorem</u> Suppose $f : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ is continuous, then for each $x_0 \in \mathbb{R}^n \in \mathcal{O}$ $\exists$ an interval $[-T,T]$ and a solution $X_t \in C^1$ of the IVP

$$\begin{cases} \dot{X}_t = f(t, X_t) & t\in[-T,T] \\ X_0 = x_0 \end{cases}$$

<u>proof</u>: We construct a sequence of approximate solutions by the Euler Method

call $K = [-a,a]\times\overline{B(x_0,r)}$ s.t. $\overline{B(x_0,r)} \subset \mathcal{O}$

Since f is cts on K it is bdd on K

There is $M>0$ s.t. $|f| \le M$ on K.

define $T = \min\left(a, \frac{r}{M}\right)$

Let $\varepsilon > 0$, $\exists$ $\delta > 0$ st. $\forall$ $(t,X), (s,Y) \in K$

w/ $|t-s| < \delta$, $|X-Y| < \delta$,

$\Rightarrow$ $|f(t,X) - f(s,Y)| < \varepsilon$

divide $[0,T]$ up into intervals ~~of length~~ $\cancel{\delta}$.

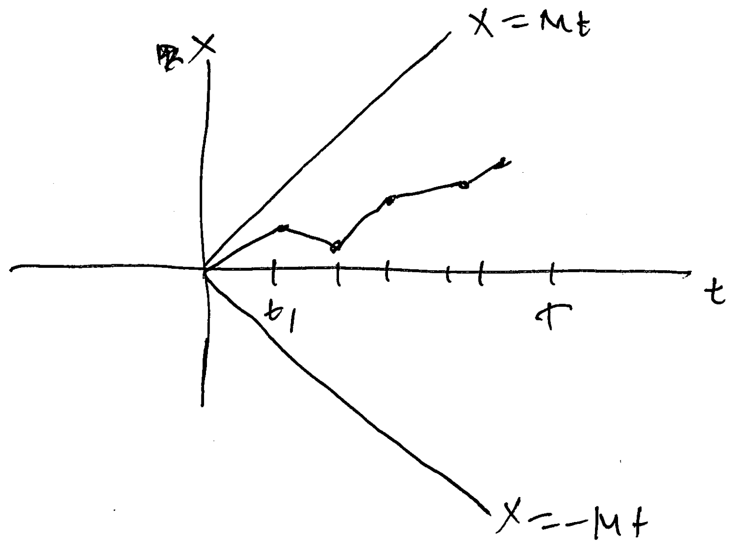
$$0 = t_0 < t_1 < \cdots < t_n = T \quad \text{s.t.}$$

$$t_i - t_{i-1} \le \min\left(\delta, \frac{\delta}{M}\right)$$

now we define a piecewise linear approximation of the ODE solution by

$$\dot{X}^\varepsilon_t = f(t_{i-1}, X^\varepsilon_{t_{i-1}}) \quad \text{for} \quad t_{i-1} < t \le t_i$$

i.e.



picture in $[0,T] \times \mathbb{R}^1$

since the slope $|\dot{X}^\varepsilon_t| = |f(t_{i-1}, X^\varepsilon_{t_{i-1}})| \le M$ for all t

(as long as $(t_{i-1}, X^\varepsilon_{t_{i-1}}) \in K$)

$$\Rightarrow \quad |X^\varepsilon_t| \le Mt \le MT \le r \quad \Rightarrow (t, X^\varepsilon_t) \in K \text{ for } t \in [0,T]$$

furthermore X^ε_t satisfies that

$$\dot{X}^\varepsilon_t = f(t_{i-1}, X^\varepsilon_{t_{i-1}}) = f(t, X^\varepsilon_t) + \left(f(t_{i-1}, X^\varepsilon_{t_{i-1}}) - f(t, X^\varepsilon_t)\right)$$

Now since $t_i - t_{i-1} \leq \frac{\delta}{M} \wedge \delta$ then

$$|X_t^\varepsilon - X_{t_{i-1}}^\varepsilon| \leq M \frac{\delta}{M} = \delta \quad \text{so}$$

$$\left| f(t_{i-1}, X_{t_{i-1}}^\varepsilon) - f(t, X_t^\varepsilon) \right| \leq \varepsilon$$

thus $\dot{X}_t^\varepsilon = f(t, X_t^\varepsilon) + \Delta_t^\varepsilon$

w/ $|\Delta_t^\varepsilon| \leq \varepsilon \quad \forall \; t \in [0,T]$ and Δ_t^ε is

~~piecewise~~ cts even (since $\Delta_{t_i}^\varepsilon = 0 \; \forall i$)

Thus X_t^ε are equicontinuous ($|\dot{X}_t^\varepsilon| \leq M$)

and uniformly bounded

$\Rightarrow \exists$ a subsequence $\varepsilon_n \to 0$ s.t.

$$X_t^{\varepsilon_n} \xrightarrow[\text{unif}]{} X_t \quad \text{on } [0,T]$$

X_t is cts (even Lipschitz w/ constant M)

$$X_t^\varepsilon = \cancel{x} \; x_0 + \int_0^t \dot{X}_s^\varepsilon \, ds = x_0 + \int_0^t f(s, X_s^\varepsilon) + \Delta_s^\varepsilon \, ds$$

now $X_s^\varepsilon \to X_t$ unif $\Rightarrow f(s, X_s^\varepsilon) \xrightarrow[\text{unif}]{} f(s, X_s)$

and $|\Delta_s^\varepsilon| < \varepsilon \Rightarrow \Delta_s^\varepsilon \to 0$ unif.

so taking limits on both sides

we are allowed to exchange limit and $\int$

by unif convergence.

$$(*) \quad X_t = x_0 + \int_0^t f(s, X_s)\, ds \qquad \forall \; t \in [0,T]$$

X_t cts $\Rightarrow \int_0^t f(s, X_s)\, ds$ is differentiable

(FTC) and even C^1

$\Rightarrow X_t$ is C^1 on $[0,T)$

now that X_t is C^1 we can ~~differentiable~~ differentiate (*) to see

$$\begin{cases} \dot{X}_t = f(t, X_t) & \text{for } t \in [0,T]. \\ X_0 = x_0 \end{cases}$$

$\square$