, section 11

methods in physical sciences part II

Subtitle: Ordinary differential equations

Instructor: Will Feldman

Office: Ryerson 360-A

Office Hours / Problem Session: TBA

-- Handout --

① A differential equation is an equation relating ~~some~~ derivatives of ~~an~~ an unknown function.

Ordinary Differential Equations involve ~~functions~~ only derivatives with respect to a single variable, i.e., unknowns are functions of 1-variable only (scalar valued).

The order of a differential equation, n, is the highest order derivative appearing in the equation.

in this course we will mostly study 1st and 2nd order differential equations.

There are good mathematical and physical reasons for this.

Linear Equations

In complete generality an ordinary differential equation of order n looks like

(1) $F(t, y, y', \ldots, y^{(n)}) = 0$ where y is ~~the~~ an unknown function ~~of $y(t)$~~ of t. and

(3) We would usually ask for (1) to hold on an interval $I = \{t : a < t < b\}$ possibly $I = (-\infty, \infty)$.

(6) Note that I didn't specify that y is a scalar or F they could be a vectors, in that case (1) would be called a <u>system of ODE's</u>.

(7) What does it mean to be a <u>solution</u> of (1) on I? A <u>solution</u> of the ODE (1) on I is a function ϕ s.t. $\phi', \ldots, \phi^{(n)}$ exist and satisfy $F(t, \phi, \ldots, \phi^{(n)}) = 0$ for $t \in I$ (give example

Linear and nonlinear differential equations

(5) We say F is <u>linear</u> if

$F(t, y, y', \ldots, y^{(n)})$. is a linear fcn of $(y, y', \ldots, y^{(n)}) \in \mathbb{R}^n$

i.e. $F(t, y, y', \ldots, y^{(n)}) = b(t) + a_0(t) y + a_1(t) y' + \cdots + a_n(t) y^{(n)}$

otherwise the ODE is <u>nonlinear</u>

<u>Fundamental Questions in study of ODE</u>

(1) Existence → is there a solution?

(2) uniqueness → is there only one solution?

(3) ~~st~~ stability → Can we construct the solution by computing numerically?

~~Ferp~~

One reason why first order ODE are important!

any nth order ODE can be changed into a 1st order system of ODE

Call $\vec{z}(t) = (y(t), y'(t), \ldots, y^{(n-1)}(t)) \in \mathbb{R}^n$

$= (z_0(t), z_1(t), \ldots, z_{n-1}(t))$

$z'(t) = (y'(t), y''(t), \ldots, y^{(n)}(t))$

$$\vec{F}(t, y, y', \ldots, y^{(n)}) = \vec{0}$$

$$\Longleftrightarrow$$

$$\vec{F}(t, z_0(t), \vec{z}'(t)) = \vec{0}$$ $\longleftrightarrow$ a first order system of ODEs.

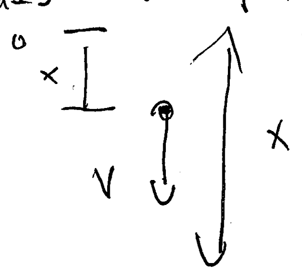
Fundamental Examples

Why are most important ODE's first or second order?

There is also a physical reason → Newton's laws.

~~Call x~~

Consider a falling object with mass m.



~~and~~ position (height) $X(t)$ $(x : \mathbb{R} \to \mathbb{R})$

and velocity $v(t)$.

Velocity is the time derivative of position

$v(t) = x'(t) = \dot{x}(t)$ we often write $\dot{x}$ when the independent variable is a time t.

acceleration is time change of velocity

~~$a(t) = v'(t)$ $\dot{x}$~~

$a(t) = \dot{v}(t) = \ddot{x}(t)$.

Newton's law ~~of~~ says that

$m a = F$ (force)

$m\dot{v} = \cancel{mg}\ -mg$ ← gravitational constant at surface of earth

this is already an ODE, but it is pretty simple we can solve by integrating

$v(t) = c - \cancel{m}gt$ (a solution for all $c \in \mathbb{R}$)

c: constant of integration determined by $v(0)$

e.g. if we dropped a ball from our hand w/ $v(0) = 0$,

$v(t) = C - \cancel{m}gt$; $v(0) = 0$

$\Updownarrow$ TBD $\Rightarrow$ $C = 0$

$v(t) = -gt$.

~~This is~~

We already see that to get a unique solution we need to specify more data. The ODE tells us how the system evolves, but it does not tell us the state of the system at any given time.

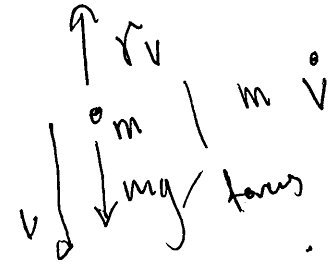
An ~~OP~~ problem of the form

$$\begin{cases} y' = f(t, y) \\ y(0) = y_0 \end{cases} \rightarrow \text{called an } \underline{\text{initial value problem}}$$

~~We can~~

~~If~~ we make our physical system a bit more complicated ~~we will already see that~~ it ~~is~~ will ~~not obvious how to~~ be a little harder to come up with a solution.

If we add in the force ~~due~~ due to drag / air resistance the ODE becomes



$m\dot{v} = -mg - \gamma v$ where γ is the "drag coefficient"

$$(2) \begin{cases} \dot{v} = -g - \dfrac{\gamma}{m} v \\ v(0) = 0 \end{cases} \rightarrow \text{Initial Value problem (IVP)}$$

$\left(g = 9.8 \dfrac{\text{meters}}{\text{second}^2} \qquad [\gamma] = \dfrac{\text{kg}}{\text{sec}} \right)$

An example from biology

~~poplut~~ population growth. (think e.g. of bacteria)

Suppose in a small time interval Δt that a fraction $r\Delta t$ of the bacteria in our colony will duplicate themselves.

Let $p(t)$ be the population at time t.

$$p(t+\Delta t) = \cancel{p(t) + r\Delta t\, p(t)} = (1 - r\Delta t)\, p(t) + 2r\Delta t\, p(t)$$
$$= p(t) + r\Delta t\, p(t)$$

rearranging

$$\frac{p(t+\Delta t) - p(t)}{\Delta t} = r\,p(t)$$ and sending $\Delta t \to 0$

$$p' = rp$$

~~$p'(t) = rp(t)$~~. an ODE model for population growth.

r called <u>growth rate</u>

$$(3) \quad \begin{cases} p' = rp \\ p(0) = p_0 \end{cases} \rightarrow \text{IVP}$$

p_0: initial population of bacteria.

now we could make the model more interesting by supposing that some scientists come and take away k bacteria per unit time for experimenting on them

$$(4) \quad \begin{cases} p' = \underbrace{rp}_{\text{pop growth}} - \underbrace{k}_{\text{constant rate of predation}} \\ p(0) = p_0 \end{cases}$$

<u>Note</u>: (4) and (2) are ~~almost~~ the <u>same</u> problem ~~at a~~ on a mathematical level.

Solving the first order linear equations,

Both models mentioned above are special cases of the general ~~ODE~~ form

$$\frac{dy}{dt} = ay - b \quad , \qquad a, b \in \mathbb{R}.$$

when $a = 0$ $\rightarrow$ just integrate

$a \neq 0$ $\rightarrow$

$$\frac{dy}{dt} = a(y - b/a)$$

if $y = \frac{b}{a}$ then $\frac{dy}{dt} = 0$

so $y = \frac{b}{a}$ (constant) is a solution

called steady state solution.

if $y \neq b/a$ then

$$\frac{dy/dt}{y - b/a} = a \qquad \text{integrate both sides}$$

$$\int \frac{dy/dt}{y - b/a}\, dt = \int a\, dt = at + C_1$$

for the left hand side use the change of variables

$t \to y(t)$

$$\int \frac{1}{y-\frac{b}{a}}\, y'(t)\, dt = \int \frac{1}{y-\frac{b}{a}}\, dy = \log\left|y-\frac{b}{a}\right| + C_2$$

so $$\log\left|y-\frac{b}{a}\right| = at + C$$

$$\left|y-\frac{b}{a}\right| = ce^{at} > 0$$

so

$$y - \frac{b}{a} = ce^{at}$$

$$\underbrace{y = \frac{b}{a} + ce^{at}}$$

family of solutions called integral curves

~~Ifs~~ for IVP $y(0) = y_0$

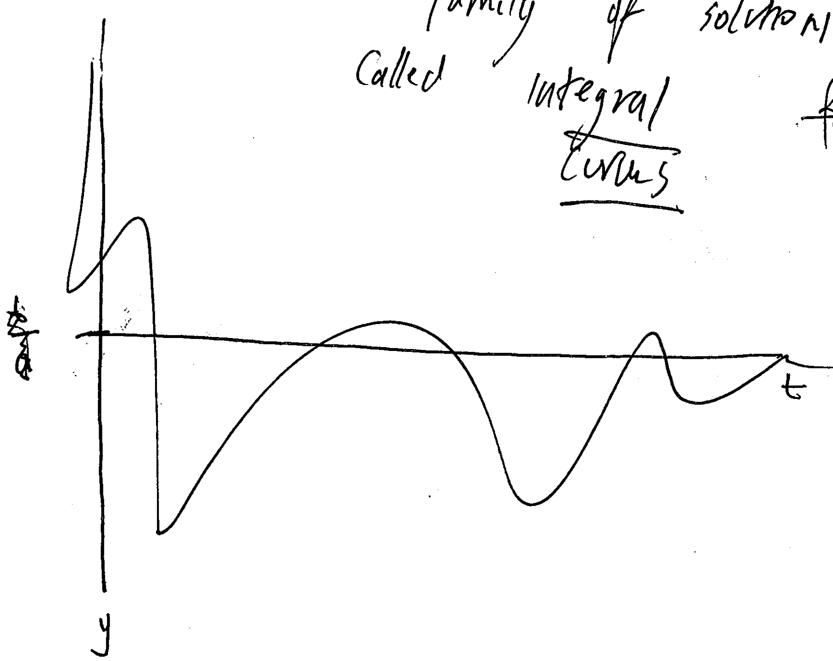
requires $y_0 = \frac{b}{a} + c$

$c = \frac{b}{a} - y_0$

~~for a = ε~~

~~b =~~

$\Rightarrow y = \frac{b}{a} + \left(\frac{b}{a} - y_0\right)e^{at}$



Example (population growth w/ constant rate of predation)

$$y' = \cancel{0.5} \; y \; \cancel{t} - 50$$

$$y(t) = \cancel{50 + (50 - y_0) e^{t}} \quad 50 + (y_0 - 50) e^{t}$$

draw

$y_0 > 50$ then y grows exponentially

if ~~y_0~~ $0 < y_0 < 50$

then $y_0 - 50 < 0$

population dies out

after a finite time.

" unstable steady state "

Example (velocity of a falling object)

$$\cancel{y' = -9.8} \quad \dot{v} = -9.8 - \cancel{0.5} \; \frac{1}{5} v$$

$$\cancel{v(t) = -19.6 + (-19.6 - v_0) e^{-0.5t}}$$

$$=$$

$$v(t) = -49 + (-49 - v_0) e^{-t/5}$$

~~if $v_0 = 0$~~

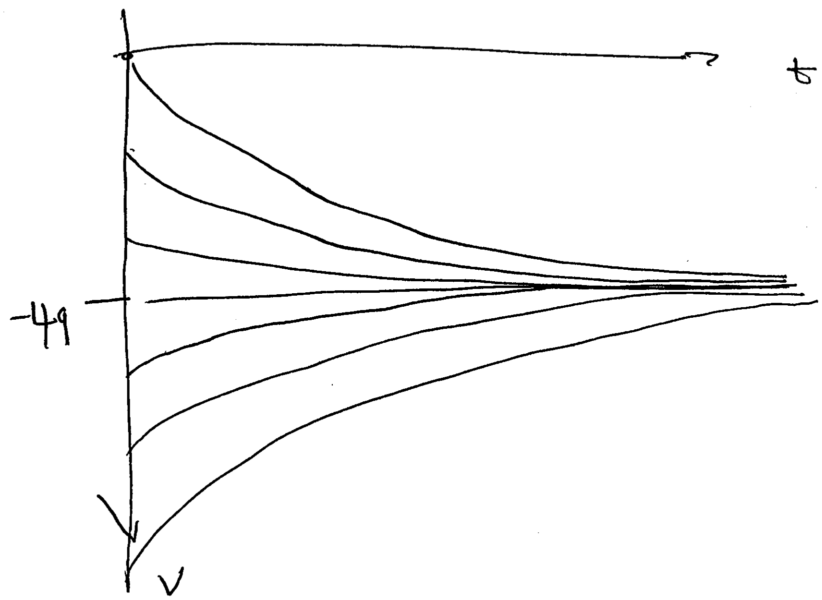
e.g. if $v_0 = 0$

$$v(t) = -49\left(1 - \underbrace{e^{-t/5}}_{\to 0 \text{ as } t \to \infty}\right)$$

similarly for other v_0

$$v(t) = -49 + \underbrace{(-49 - v_0)\, e^{-t/5}}_{\to 0 \text{ as } t \to \infty}$$

so $v(t) \to -49$ (the $\underbrace{\text{steady state solution}}_{\text{called "terminal velocity"}}$)



"stable steady state"

the <u>method of integrating factors</u>

More general than $y' = ay + b$ is

$$\frac{dy}{dt} + p(t)y = g(t) \quad \text{or} \quad P(t)\frac{dy}{dt} + Q(t)y = G(t)$$

Still <u>linear</u> equation but coeff

if $P(t) \neq 0$ ever then these are the same type of equation.

at the moment we can't solve in this generality,

but what if

$Q(t) = P'(t)$? then

$$\underbrace{P(t)\frac{dy}{dt} + P'(t)y}_{} = G(t)$$

$$\left(\text{~~chain~~ product rule}\right) \quad \frac{d}{dt}\left(P(t)y\right) = G(t)$$

~~so $P(t)y$~~

$$P(t)y(t) - P(0)y(0) = \int_0^t G(s)\,ds$$

$$\underbrace{y(t) = \frac{P(0)}{P(t)}y(0) + \frac{1}{P(t)}\int_0^t G(s)\,ds}_{\text{great, we solved it.}}$$

The trick of the integrating factor method is that we can multiply the ~~eq~~ ODE by a factor $\mu(t)$ to make it integrable.

$$\frac{dy}{dt} + p(t)\,y = g(t)$$

multiply by $\mu(t)$

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)\,y = \mu(t)g(t)$$

$\cancel{\frac{dy}{dt}}$ need $\mu(t)p(t) = \mu'(t)$ ~~for~~ so that

$$LHS = \frac{d}{dt}\left[\mu(t)\,y\right] = \mu(t)\,g(t)$$

this is just the homogeneous linear ODE

w/ same rate $p(t)$.

can solve by integrating.

~~then~~

$$\frac{\mu'(t)}{\mu(t)} = p(t)$$

$\downarrow \int$ both sides w.r.t. t

$$\int \frac{d\mu}{\mu} = \int p(t)\,dt + k \quad \swarrow \text{ integrating factor}$$

} any such soln works
take $k=0$

$$\log|\mu| = \int p(t)\,dt \;\cancel{+k}$$

$$\mu = \exp\left(\int p(t)\,dt\right)$$

now from the choice of $\mu(t)$,

$$\frac{d}{dt}\left[\exp\left(\int p(t)\,dt\right) y\right] = \exp\left(\int p(t)\,dt\right) g(t)$$

~~exp~~

$$\mu(t) y = \int \mu(s) g(s)\,ds + k$$

$$y = \frac{1}{\mu(t)}\left[\int \mu(s) g(s)\,ds + k\right]$$

for example applying the method when

$$\frac{dy}{dt} + ay = g(t)$$

need $\mu' = a\mu$

so $\int \frac{d\mu}{\mu} = at$

$\log|\mu| = at$

$\mu = e^{at}$

$$\frac{d}{dt}\left[e^{at} y\right] = e^{at} y' + a e^{at} y = e^{at} g(t)$$

So $e^{at} y = \int e^{as} g(s)\,ds + k$

and

$$y = ke^{-at} + \int e^{-a(t-s)} g(s)\, ds$$

Tip: It is better to just remember the method than the formula

(1) multiply equation by $\mu(t)$

(2) set $LHS = \frac{d}{dt}[\mu(t) y] = \mu y' + \mu' y$

leads to ODE for μ.

Example Solve the IVP

$$\begin{cases} 2y' + ty = 2 \\ y(0) = 1 \end{cases}$$

multiply by int factor

$$\mu(t) y' + \frac{t}{2} \mu(t) y = \mu(t)$$

set

$$\mu'(t) = \frac{t}{2} \mu(t)$$

$$\frac{\mu'}{\mu} = \frac{t}{2} \rightarrow \int_{\mu(0)}^{\mu(t)} \frac{d\mu}{\mu} = \int_0^t \frac{s}{2}\, ds$$

$$\log|\mu(t)| - \log|\mu(0)| = \frac{t^2}{4}$$

$$\mu(t) = \mu(0) e^{t^2/4}$$

take $\mu(0) = 1$

Then

$$(e^{t^2/4}y)' = e^{t^2/4}$$

$$\int_0^t (e^{t^2/4}y)' = \int_0^t e^{s^2/4}\,ds$$

$$e^{t^2/4}y - e^{0/4}y(0) = \int_0^t e^{s^2/4}\,ds \quad , \quad y(0)=1$$

$$y = e^{-t^2/4} + \underbrace{\int_0^t e^{s^2/4 - t^2/4}\,ds}$$

can evaluate numerically, but not analytically.

<u>Note</u>: When we solve

(*) $y' + p(t)y = g(t)$

the equation for the integrating factor is always

(**) $\mu' = p(t)\mu$ this is the <u>homogeneous</u> version of eqn above with a minus sign.

Note that if μ solves (**) then $\phi(t) = \mu(-t)$ solves $\phi' = -\mu'(-t) = -p(-t)\phi(t)$

2.2 Separable Equations

In this section we will call the independent variable x, and look at

$$\frac{dy}{dx} = f(x,y) \quad , \quad \text{(we will need } f \text{ later for something)}$$

We rewrite this as

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

Now we say that the ODE is <u>separable</u> if $M(x)$ only and $N(y)$ only depend on x, y respectively

$$M(x) + N(y)\frac{dy}{dx} = 0 \qquad (1)$$

(since formally this is $M(x)dx + N(y)dy = 0$) in the sense that if $(x, y(x))$ ~~$x \in I$~~ ~~$y(x)$~~ is an integral curve of (1) then

line integral $\displaystyle\int_\gamma M(x)dx + N(y)dy = \int_I \left[M(x) + N(y)\frac{dy}{dx}\right]dx = \int_I 0\,dx = 0$

Essentially we want to view

$(M(x), N(y))$ as the gradient of some

$F(x,y)$ since then

for any $y(x)$ satisfying (1)

$$\frac{d}{dx} F(x, y(x)) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

$$= M(x) + N(y)\frac{dy}{dx} = 0$$

i.e. $F(x, y(x)) = c$ along any integral curve.

To solve an IVP

$$(2) \quad \begin{cases} M(x) + N(y)\frac{dy}{dx} = 0 \\ y(x_0) = y_0 \end{cases}$$

Integrate in each coordinate

$$F(x,y) = \int_{x_0}^{x} M(s)\,ds + \int_{y_0}^{y} N(s)\,ds$$

this satisfies that $\nabla F = (M, N)$

and also since F constant on

the solution of the IVP and

$F(x_0, y_0) = 0$ $\Rightarrow$ the integral curve

associated w/ IVP (2) is contained

in the 0 - level set of

$$F(x,y) = \int_{x_0}^{x} M(s)\,ds + \int_{y_0}^{y} N(s)\,ds$$

This is an implicit rep'n of the solution (but still quite useful)

Example

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{3x^2+4x+2}{2(y-1)} \\ y(0) = -1 \end{cases} \rightarrow -(3x^2+4x+2) + 2(y-1)\frac{dy}{dx} = 0$$

$$F(x,y) = \int_0^x -(3s^2+4s+2)\,ds + \int_{-1}^{y} 2(s-1)\,ds$$

$$= -x^3 - 2x^2 - 2x + y^2 - (-1)^2 - 2(y+1)$$

$$= -x^3 - 2x^2 - 2x - 3 + y^2 - 2y$$

In this case we can solve quadratic for

$$y = 1 \pm \frac{1}{2}\left(4+4\left(x^3+2x^2+2x+3\right)\right)^{1/2}$$

$$= 1 \pm \left(x^3+2x^2+2x+4\right)^{1/2}$$

two possible solutions, we need

$$-1 = y(0) = 1 \pm (4)^{1/2} = 1 \pm 2$$

so need the $(-)$ sign

$$y(x) = 1 - \left(x^3+2x^2+2x+4\right)^{1/2}$$

notice that this is only a valid solution

when $\quad x^3+2x^2+2x+4 > 0$

~~i.e. x~~

when $\quad x \ll 0 \quad$ is neg

$x \geq 0$ ~~$x \gg 0$~~ $\quad$ is pos

can check $\quad (x+2)(x^2+2) = x^3+2x^2+2x+4$

only one real zero at $x=-2$.

so solution is valid for $x > -2$.

Sometimes you can take a ~~nonlinear~~ system and turn it into a single separable equation

e.g.

$$\begin{cases} \dfrac{dx}{dt} = -y \\ \dfrac{dy}{dt} = x \end{cases} \quad \text{w/} \quad (x(0), y(0)) = \langle 1, 0 \rangle$$

(solution is $(\cos t, \sin t)$ you can check)

but we can formally write

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{x}{-y} \quad \rightarrow \quad x + y\frac{dy}{dx} = 0$$

$$F(x,y) = \int_1^x s\,ds + \int_0^y s\,ds$$

$$= \frac{x^2}{2} - \frac{1}{2} + \frac{y^2}{2}$$

So level curve $F(x,y) = 0$

$$\Updownarrow$$

$x^2 + y^2 = 1$ is ~~solution~~ integral curve for the separable ODE.

Notice that we lost information about t,

but at least we know shape of level curves

Ex (more complicated)

$$\frac{dx}{dt} = -x^2(y-4)$$

$$\frac{dy}{dt} = x^2(x+2)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{x^2(x+2)}{-x^2(y-4)} = \frac{x+2}{-(y-4)}$$

$$x+2 + (y-4)\frac{dy}{dx} = 0$$

$$(x+2)\,dx + (y-4)\,dy = 0.$$

Separable equation

2.6 Exact Equations

Recall we discussed earlier equations of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \qquad (1)$$

We showed that if we can find $\psi(x,y)$ so that $M = \frac{\partial\psi}{\partial x}$, $N = \frac{\partial\psi}{\partial y}$

then the ODE becomes

$$\frac{d}{dx}\left[\psi(x,y(x))\right] = 0$$

So integral curves of (1) lie on level curves of ψ, and

$\psi(x,y(x)) = c$ ~~can~~ implicit eqn for $y(x)$ can be solved for y.

If such a ψ exists the ODE (1) is said to be *exact*.

when $M(x)$, $N(y)$ only ~~this~~ it was easy to show that such a ψ existed.

What about in greater generality? When ~~can~~ can we integrate (M, N) to get ψ?

Theorem ~~Let~~ Suppose $M, N, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ are continuous in the rectangle

$$R = (\alpha, \beta) \times (\gamma, \delta).$$ Then (1)

~~has~~ is *exact* if and only if

$$M_y = N_x \quad \text{at each point of } R,$$

i.e. there exists ψ such that

$$\psi_x = M, \quad \psi_y = N.$$

first note that $M_y = N_x$ is ~~de~~ a necessary condition recall that for C^2 ψ

$$\psi_{xy} = \psi_{yx} \qquad \text{(equality of mixed partials)}$$

So in order for $M = \psi_x$, $N = \psi_y$ must have $M_y = \psi_{xy} = \psi_{yx} = N_x$.

~~On the other hand it turns~~

For the other direction we integrate in x to get

$$\psi(x,y) = Q(x,y) + h(y)$$

where

$$Q(x,y) = \int_{x_0}^{x} M(s,y)\,ds$$

and $h(y)$ is a constant of integration.

Now
$$\psi_y = \frac{\partial Q}{\partial y} + h'(y)$$
$$= \int_{x_0}^{x} \frac{\partial M}{\partial y}(s,y)\,ds + h'(y)$$

should be $= N(x,y)$

solving for the unknown h' yields

$$h'(y) = N(x,y) - \int_{x_0}^{x} \frac{\partial M}{\partial y}(s,y)\,ds$$

now in order for this to be possible the RHS (despite appearances) cannot depend on x, i.e.

$$0 = \frac{\partial}{\partial x}\left(h'(y)\right) = \frac{\partial}{\partial x} N(x,y) - \frac{\partial}{\partial x}\int_{x_0}^{x} \frac{\partial M}{\partial y}(s,y)\,ds$$

$$= \frac{\partial N}{\partial x}(x,y) - \frac{\partial M}{\partial y}(x,y)$$

which is true by assumption of the theorem.

integrating in y

$$h(y) = \int_{y_0}^{y} N(x,t) - \frac{\partial Q}{\partial y}(x,t)\,dt$$

$$= \int_{y_0}^{y}\left[N(x,t) - \int_{x_0}^{x} \frac{\partial M}{\partial y}(s,t)\,ds\right] dt$$

Example ~~$\psi(x,y)=$~~

$$(y\cos x + 2xe^y)\,dx + (\sin x + x^2e^y - 1)\,dy = 0$$

$$\frac{\partial M}{\partial y} = \cos x + 2xe^y \qquad \frac{\partial N}{\partial x} = \cos x + 2xe^y$$

$\Rightarrow$ Eqn is exact

$$\psi(x,y) = \int y\cos x + 2xe^y\,dx + h(y)$$

$$= y\sin x + x^2e^y + h(y)$$

$$N(x,y) = \frac{\partial \psi}{\partial y} = \sin x + x^2e^y + h'(y) \Rightarrow h'(y) = -1$$

so $h(y) = -y + c$ can take $c=0$

$$\psi(x,y) = y\sin x + x^2e^y - y$$

(dumb) Integrating factors This trick sometimes (read: never) works

try multiplying by $\mu(x,y)$ to make the equation exact.

$$\mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y)\,dy = 0$$

Need $(\mu M)_y = (\mu N)_x$

i.e.

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

$$\mu_x M - \mu_y N + \mu (M_y - N_x) = 0$$

any μ satisfying this condition will work, there may be many solutions.

Unfortunately this equation is not really easier than before (in fact it is a PDE now not an ODE)

We could look for μ independent of y or x to make matters simpler. ~~The~~ If we look for $\mu(x)$, then $\mu_y = 0$ so need

$$\mu_x M + \mu (M_y - N_x) = 0$$

or $\quad \mu_x = \frac{M_y - N_x}{N}\mu$

now if $\frac{M_y - N_x}{N}$ is independent of y

then this equation is just

$\mu' + p(x)\mu = 0$ which we can

solve by method of integrating factors.

<u>Example</u> $(3xy + y^2) + (x^2 + xy)y' = 0$

$M_y = 3x + 2y$

$N_x = 2x + y$

$M_y - N_x = x + y$

$\frac{M_y - N_x}{N} = \frac{\cancel{x+y}}{x\cancel{(x+y)}} = \frac{1}{x}$

depends on x only

so solving $\mu' = \frac{1}{x}\mu$ get $\mu(x) = x$

so multiply the ODE by x to make it exact.

2.3 Modeling w/ 1^{st} order Eqns

Read the chapter (2.3)

Example 1: (Pay attention to units)

A time $t=0$ a tank contains Q_0 lb of salt dissolved in 100 gal of water. Water containing $\frac{1}{4}$ lb salt / gal is entering the tank at a rate of r gal/min and the well-stirred mixture leaves the tank at the same ~~same~~ rate.

(1) set up IVP

(2) What is the limiting amount of salt Q_L as $t \to \infty$?

(3) how long must we wait till

$$\left|\frac{Q}{100} - \frac{Q_L}{100}\right| \le .02 \quad w/ \quad r=3, \; Q_0 = 2Q_L$$

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out} \qquad \text{both in } \frac{lb}{min}$$

$$[\text{rate in}] = \left[\frac{lb}{gal}\right] \cdot \left[\frac{gal}{min}\right] \quad \to \quad \text{rate in} = \frac{1}{4}\frac{lb}{gal} \cdot r\frac{gal}{min}$$

$$= \frac{r}{4}\frac{lb}{min}$$

rate out = concentration of water in tank $\cdot$ $\frac{\text{gal}}{\text{min}}$ out of mixture

$$= \frac{Q(t)\ \text{lb}}{100\ \text{gal}} \cdot r\ \frac{\text{gal}}{\text{min}} = \frac{rQ(t)}{100}\ \frac{\text{lb}}{\text{min}}$$

so $\begin{cases} \dfrac{dQ}{dt} = \dfrac{r}{4} - \dfrac{rQ}{100} \\ Q(0) = Q_0 \end{cases}$ is IVP for the salt concentration

~~there~~ is an equilibrium when

$$0 = \frac{r}{4} - \frac{rQ_E}{100} \Rightarrow \frac{1}{4} - \frac{Q_E}{100} = 0$$
$$\Rightarrow Q_E = 25$$

is $Q_L = Q_E$? physical reasoning says yes.

$$\frac{d}{dt}(Q - Q_E) = r\left(\frac{1}{4} - \frac{Q}{100}\right) = r\left(\frac{1}{4} - \frac{Q - Q_E}{100} - \frac{Q_E}{100}\right)$$
$$= r\left(\overbrace{\frac{1}{4} - \frac{Q_E}{100}}^{0} - \frac{Q - Q_E}{100}\right)$$

$$\frac{d}{dt}(Q - Q_E) = -\frac{r}{100}(Q - Q_E)$$

so $Q - Q_E = (Q_0 - Q_E)\, e^{-\frac{r}{100}t} \to 0$ as $t \to \infty$.

i.e. $Q_L = Q_E = 25$ lbs

w/ $r = 3$, $Q_0 = 50$ lbs

$$Q - Q_E = 25 e^{-\frac{3}{100}t}$$ monotone decreasing

$$25 e^{-\frac{3}{100}t} = 2$$ when

$$e^{-\frac{3}{100}t} = \frac{2}{25}$$

$$-\frac{3}{100}t = \log \frac{2}{25}$$

$$t = \frac{100}{3} \log \frac{25}{2}.$$

__Example 2__ Bank account.

Suppose I put S_0 dollars in a bank account w/ interest compounded continuously at r % per year

$$\frac{dS}{dt} = rS$$

$$S(t) = e^{rt} S_0$$

S_0 ← initial deposit.

If interest is instead compounded, say, m times per year then

$$S\left(\frac{k}{m}\right) = S\left(\frac{k-1}{m}\right) + \frac{r}{m} S\left(\frac{k-1}{m}\right)$$

$$= \left(1+\frac{r}{m}\right) S\left(\frac{k-1}{m}\right) = \cdots = \left(1+\frac{r}{m}\right)^k S_0$$

so $$S(t) = \left(1+\frac{r}{m}\right)^{mt}$$
$t = \frac{k}{m}$

as $m \to \infty$ for fixed t

$$\lim_{m\to\infty} S_0\left(1+\frac{r}{m}\right)^{mt} = e^{rt} \cdot S_0$$

2.4 Linear and Nonlinear Equations

(Existence & uniqueness)

Consider our problem from before

$$(*) \begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases}$$

We already showed that $(*)$ has a solution (Existence) by the method of integrating factors.

but is this the only possible solution? Maybe a different method would yield a different solution?

<u>Thm</u> If p, g are continuous on $I = (\alpha, \beta) \subseteq \mathbb{R}$, $t_0 \in I$ then for any $y_0 \in \mathbb{R}$ there exists a <u>unique</u> function $y = \phi(t)$ satisfying the IVP (*).

Actually we proved this by the construction of the solution. For more general Nonlinear equations the result is more delicate.

$$(**) \quad \begin{cases} y' = f(t, y) & \text{in } I = (\alpha, \beta) \ni t_0 \\ y(t_0) = y_0 \end{cases}$$

Theorem Suppose f, $\frac{\partial f}{\partial y}$ are continuous in

the a rectangle $I \times (\gamma, \delta)$ containing (t_0, y_0).

Then in some interval $t_0 - h < t < t_0 + h$

contained in $\alpha < t < \beta$ there is

a unique solution $y = \phi(t)$ of the

IVP (**).

(Counter) - Examples (why are hypotheses of the theorem "necessary")

$$\begin{cases} y' = y^2 \\ y(0) = 1 \end{cases} \rightarrow \int_1^y \frac{dy}{y^2} = \int_0^t 1 \, dt$$

$$-\frac{1}{y} + 1 = t$$

$$y = \frac{-1}{t-1} = \frac{1}{1-t}$$ has singularity

at $t = 1$ so it can't solve the

ODE there.

i.e. solution may only exist on a short time

interval

$$\begin{cases} y' = y^{\frac{1}{2}} \\ y(0) = 0 \end{cases}$$

~~integrate~~ $\int$

$$\int_0^y \frac{dy}{y^{1/2}} = \int_0^t 1\, ds$$

~~$y(t) =$~~

$$2 y^{1/2} = t$$

$$y = \cancel{\left(\tfrac{2}{3}t\right)^{\frac{3}{2}}} \left(\frac{t}{2}\right)^2$$

$y(t) = \dfrac{t^2}{4}$ solves.

<u>but</u> also
$$\phi(t; t_0) = \begin{cases} \dfrac{(t-t_0)^2}{4} & t \geq t_0 \\ 0 & t \leq t_0 \end{cases}$$
for all $t_0 > 0$

$\phi(t; t_0)$ ~~are~~ all satisfy $\phi(0; t_0) = 0$

And they are all C^1 fns which satisfy the ODE for all $t \in \mathbb{R}$.

the problem here is that

$f(t, y)$ is ~~are~~ not C^1 at $(t_0, 0)$

$$\left(\frac{\partial f}{\partial y} = \frac{1}{y^{1/2}} \right)$$

This example is not just fabricated by the way. Non-uniqueness actually happens in ~~the~~ models coming from physical problems. Just needs to be interpreted correctly.

2.5 Autonomous Equations

$y' = f(t,y)$ Called autonomous if f is indep of t.

i.e.

$y' = f(y)$.

2.5 Autonomous Equations and Population Dynamics

$$\frac{dy}{dt} = f(y) \quad \text{w/ } f \text{ independent of } t$$

called ~~auto~~ **autonomous**

Exponential growth

$$\begin{cases} \frac{dy}{dt} = ry \\ y(0) = y_0 \end{cases} \qquad y(t) = y_0 e^{rt}$$

this is a reasonable model for populations in some situations (no limitation on growth like predation or food supply)..

it may be more reasonable for the growth rate to depend on the total population size

$$\frac{dy}{dt} = h(y)\,y \qquad \text{where } h(y) \approx r \text{ when } y \text{ small}$$
$$\text{and } h(y) < 0 \text{ when } y \text{ very large}$$

A simple function having these properties

$h(y) = r - ay \qquad a > 0$

which is usually written

$$\begin{cases} \dfrac{dy}{dt} = r\left(1 - \dfrac{y}{K}\right)y \\ y(0) = y_0 \end{cases}$$

$K = \frac{r}{a}$ carrying capacity

r intrinsic growth rate

It is often useful to first get a good qualitative idea about the behavior of solutions.

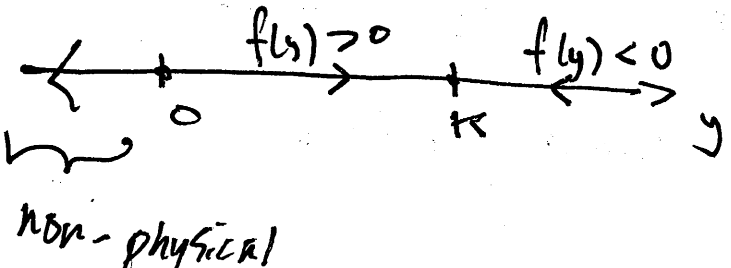
First we locate the equilibria / steady states / critical points of f

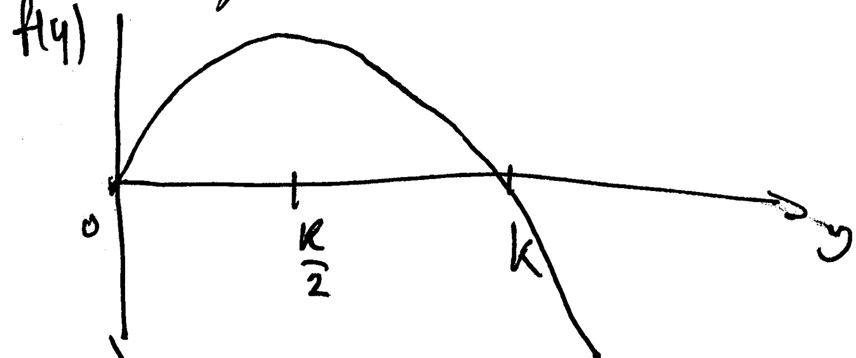
$0 = f(y) = r\left(1 - \frac{y}{K}\right)y$

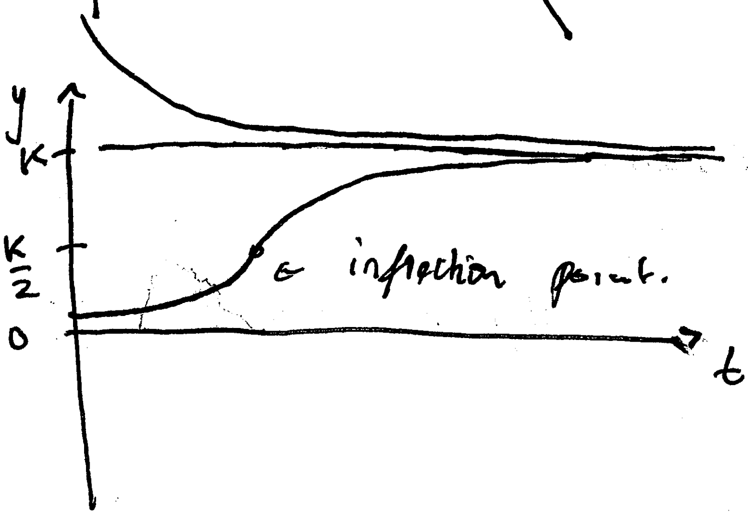
when $y = K$ or $y = 0$

Next it is useful to draw the

<u>phase line</u>







$\frac{d^2y}{dt^2} = f'(y)\,y'(t)$

for $0<y<K$

$y' = f(y) > 0$

so $y'' = f'(y)\,y'$

$\begin{cases} y''>0 & y<\frac{K}{2} \\ y''<0 & y>\frac{K}{2} \end{cases}$

if $y>K$ $y'<0$, $f'(y)<0$

so $y''>0$ ~~concave~~ convex

By uniqueness theorem solution cannot cross

so if $0<y_0<K \Rightarrow 0<y(t)<K$ for all $t\in\mathbb{R}$

similarly if $y_0 > K \Rightarrow y(t) > K$

K is called <u>carrying capacity</u>

~~&~~ $y(t) \to K$ as $t \to \infty$

but is never actually equal.

Exact solution of

$$\begin{cases} \frac{dy}{dt} = +r\,(1 - y/K)\,y \\ y(0) = y_0 \end{cases}$$

this equation is separable

$$\frac{1}{(1-y/K)y}\frac{dy}{dt} = +r \qquad \text{integrating between } 0 \text{ and } t, \text{ as}$$

$$\int_{y_0}^{y(t)} \frac{dy}{(1-y/K)y} = \int_0^t r\,dt$$

use partial fractions on the left

$$\frac{1/K}{1-y/K} + \frac{1}{y} = \frac{y/K + 1 - y/K}{(1-y/K)y} = \frac{1}{(1-y/K)y}$$

So we get

$$\log|y| - \log|1-y/K| - (\log|y_0| - \log|1-y_0/K|) = rt$$

$$\frac{y}{1-y/K}\,\frac{1-y_0/K}{y_0} = e^{rt}$$

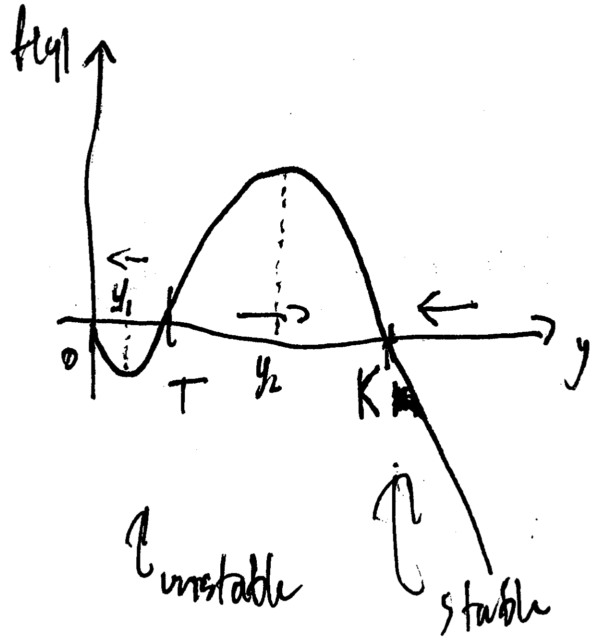
$$y = (1-y/K)\,\frac{y_0}{1-y_0/K}\,e^{rt}$$

$$y\left(1+\frac{1}{K}\,\frac{y_0}{1-y_0/K}\,e^{rt}\right) = \frac{y_0}{1-y_0/K}\,e^{rt}$$

$$y(t) = \frac{\frac{y_0}{1-y_0/K}\,e^{rt}}{1+\frac{1}{K}\,\frac{y_0}{1-y_0/K}\,e^{rt}}$$

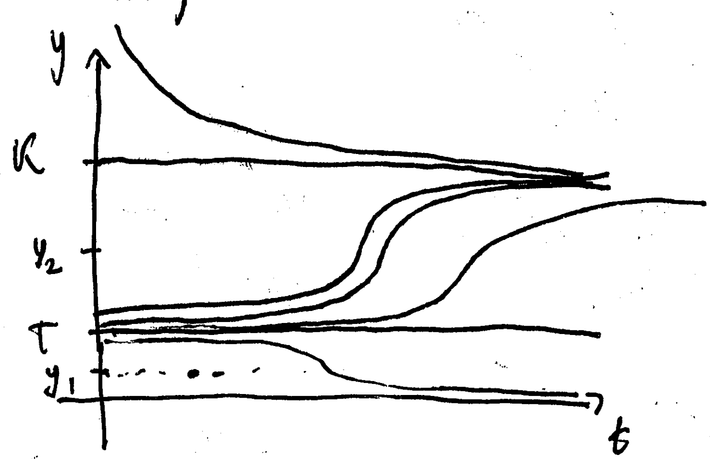
$$= \frac{y_0}{(1-y_0/K)e^{-rt} + y_0/K} = \frac{y_0 K}{(K-y_0)e^{-rt}+y_0}$$

Example $\frac{dy}{dt} = -r\underbrace{(1-\frac{y}{T})(1-\frac{y}{K})y}_{\text{zeros at } y=0,\ T,\ K}$ $\qquad 0<T<K<\infty$



y_1, y_2 are where inflections of $y(t)$ are.

drawing



From the drawing it looks like solutions in each region are just shifts of each other.

This is true.

$$\begin{cases} \frac{dy_1}{dt} = f(y_1) \\ y_1(0) = y_{10} \end{cases}, \quad \begin{cases} \frac{dy_2}{dt} = f(y_2) \\ y_2(0) = y_{20} \end{cases}$$

suppose wlog that $T < y_{10} < y_{20} < K$

by my theorem on the next page of notes

$\lim_{t \to \infty} y_1(t) = K$ so and y_1 is cts so

there is $t_* > 0$ so that

$y_1(t_*) = y_{20}$

then $y_1(t + t_*) = z(t)$ Claim: $z(t) = y_2(t)$

$$\begin{cases} \frac{dz}{dt} = f(z(t)) \\ z(0) = y_1(t_*) = y_{20} \end{cases}$$, uniqueness $\Rightarrow z(t) = y_2(t)$

so $y_2(t) = y_1(t + t_*)$.

Suppose f is such that the uniqueness theorem holds

$$\begin{cases} \frac{dy}{dt} = f(y) \\ y(0) = y_0 \end{cases} , \quad \cancel{f(y_0) > 0}$$

$f(y) > 0$ in (a,b), $f(a) = f(b) = 0$

$y_0 \in (a,b)$

What is $\lim_{t \to \infty} y(t)$?

since $\frac{dy}{dt} = f(y) \Rightarrow y(t) = \underbrace{y_0 + \int_0^t f(y(s))\,ds}$

continuous function of t

~~So~~ Claim 1: $y(t) \in (a,b)$ for all $t \in \mathbb{R}$

Suppose otherwise, by continuity $\exists$ $t_* \in \mathbb{R}$

so that $y(t_*) \in \{a, b\}$ since the equation is autonomous

$z(t) = y(t + t_*)$ solves

$$\begin{cases} z'(t) = f(z(t)) \\ z(0) = a \quad (\text{or } b) \end{cases}$$

and therefore $z(t) \equiv a$ (or b)

contradicts $y_0 \in (a,b)$. $\square$

~~Claim 2~~: ~~Suppose~~

from Claim 1 $y'(t) = f(y) > 0$ for all t

(since $y(t) \in (a,b)$ for all $t \in \mathbb{R}$

so $y(t)$ is mono-tone increasing function of t,

and therefore

$$\lim_{t\to\infty} y(t) = y_\infty \quad \text{exists}$$

suppose $y_\infty < 6$, ~~then~~

Then ~~let~~ $c = \min_{y\in[y_0, y_\infty]} f(y) > 0$

and so $y'(t) = f(y(t)) \geq c > 0$ for all $t>0$

$\Rightarrow$ $y(t) \geq y_0 + ct$ (integrating both sides)

$\Rightarrow$ $y\left(\frac{1}{c}(6-y_0)\right) = 6 > y_\infty$ ~~$\Rightarrow$~~

Contradicts y monotone increasing

w/ $\lim_{t\to\infty} y(t) = 6$.

$\Rightarrow$ $y_\infty \geq 6$ $\Rightarrow$ $y_\infty = 6$

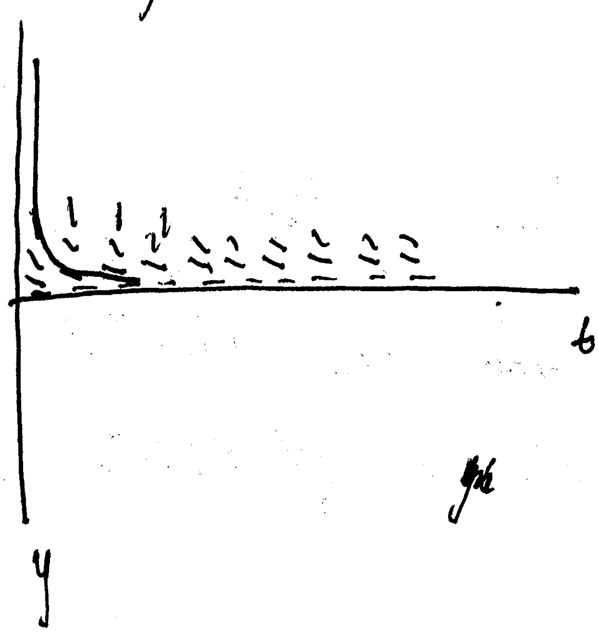
2.7 Euler's Method

In practice ODE's are solved numerically.

$$\begin{cases} \frac{dy}{dt} = f(t,y) \\ y(t_0) = y_0 \end{cases}$$ initial value problem.

except for very special situations the methods we have learned so far fail

Solving using Euler's method is essentially equivalent to constructing integral curves using a direction field.



(e.g.)

$f(t,y) = -y$

for t very small

$$y(t) - y(0) = \int_{t_0}^{t} \frac{dy}{dt}\, ds = \int_{t_0}^{t} f(s, y(s))\, ds$$

$$\approx f(t_0, y_0)(t - t_0)$$

i.e.

$$y(t) \approx y_0 + f(t_0, y_0)(t - t_0)$$

Only valid for t very small

but we can iterate,

take h small and define

$$y_1 = y_0 + f(t_0, y_0)h \qquad t_1 = t_0 + h$$

$$y_2 = y_1 + f(t_1, y_1)h \qquad t_2 = t_1 + h$$

$$\vdots$$

$$y_k = y_{k-1} + f(t_{k-1}, y_{k-1})h \qquad t_k = t_{k-1} + h$$

This is the most basic numerical method to construct solutions of ODE's.

2.8 Existence and Uniqueness

$$(1)\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$$

is there ~~an~~ a solution?
is there only one?

Thm If f and $\frac{\partial f}{\partial y}$ are ~~cts~~ continuous in a rectangle R : ~~[-a, a]~~ $|t| \le a$, $|y| \le b$, then there is some time interval $|t| \le h \le a$ in which there exists a unique solution $y = \phi(t)$ of the IVP (1).

The proof is more convenient in the integral form of the ODE.

If we suppose we have a solution ϕ

then $\phi'(t) = f(t, \phi(t))$

$\underline{\qquad\qquad}$ continuous function $\forall\, t$

so we can integrate to get

$$(2) \quad \boxed{\phi(t) = \phi(t_0) + \int_{t_0}^{t} f(s, \phi(s))\, ds} \qquad \text{Integral} \; \cancel{\text{eqn}}$$

note that the initial value is already included in (2).

This is a bit easier to work with since anti-derivatives ~~are nicer than~~ make functions smoother ~~while derivatives~~

to construct a solution we make successive approximations.

$$\phi_0(t) = \cancel{\phi(t_0)} = y_0$$

$$\phi_1(t) = y_0 + \int_{t_0}^{t} f(s, \phi_0(s))\, ds$$

$$\vdots$$

$$\phi_{k+1}(t) = y_0 + \int_{t_0}^{t} f(s, \phi_k(s))\, ds .$$

we can identify

$$\phi_n(t) = \phi_0(t) + [\phi_1(t) - \phi_0(t)] + \cdots + [\phi_n(t) - \phi_{n-1}(t)]$$

that
so we could identify the limit $\phi(t) = \lim_{n\to\infty} \phi_n(t)$
if the infinite sum

$\sum_{k=1}^{\infty} [\phi_k - \phi_{k-1}](t)$ converges.

to check this we estimate

$$|\phi_k(t) - \phi_{k-1}(t)| = \left|\int_{t_0}^{t} f(s, \phi_{k-1}(s)) - f(s, \phi_{k-2}(s))\, ds\right|$$

$$\leq \int_{t_0}^{t} |f(s, \phi_{k-1}(s)) - f(s, \phi_{k-2}(s))|\, ds$$

Since $\frac{\partial f}{\partial y}$ cts on $R = [-a, a] \times [-b, b] + (t_0, y_0)$

$\Rightarrow$ $|f|, \left|\frac{\partial f}{\partial y}\right| \leq M$ on R for some $M < \infty$.

since $|\phi_k'| \leq M$ as long as (t, ϕ_k) is in R

we know $|\phi_k - y_0| \leq M|t - t_0|$

so $(t, \phi_k) \in R$ as long as $M|t - t_0| \leq b$

true for all k so

$$|\phi_k(t) - \phi_{k-1}(t)| \leq M|t - t_0| \max_{0 \leq s \leq t - t_0} |\phi_{k-1}(t_0+s) - \phi_{k-2}(t_0+s)|$$

i.e. $$\max_{|t-t_0|\le T} |\phi_k - \phi_{k-1}| \le MT \max_{|t-t_0|\le T} |\phi_{k-1} - \phi_{k-2}|$$

$$\vdots$$

$$\le (MT)^{k-1} \max_{|t-t_0|\le T} |\phi_1 - \phi_0|$$

since T suff small s.t. $|\phi_1 - y_0| \le b$ by assumption above

$\forall k$

$$\Rightarrow \max_{|t-t_0|\le T} |\phi_k - \phi_{k-1}| \le (MT)^{k-1} b$$

if T suff small so that

$$MT \le \frac{1}{2}$$

then sum is dominated by geometric series so it converges.

Chapter 3 Second order Linear ODE

3.1 Homogeneous Equations w/ constant coefficients

(1) $$\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt})$$

equation (1) is called linear if

$$f(t, y, \frac{dy}{dt}) = g(t) - p(t)\frac{dy}{dt} - q(t)y$$

g, p, q depend on t but not y.

Which we write

(2) $$y'' + p(t)y' + q(t)y = g(t)$$

or

(3) $$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

Note: prove that linear combination of solutions is a solution.

An initial value problem for (1), (2) or (3)

would consist of the ODE along w/ initial conditions for y, y',

$$\begin{cases} y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$$

This is reasonable to expect since, for example, to solve

$y'' = 0$ uniquely need two initial datas for $y(0)$, $y'(0)$

$\downarrow$

$$y(t) = \frac{t^2}{2} + At + B$$

"two integrations $\to$ two constants of integration"

A second order equation is said to be homogeneous if $g(t) = 0$ in (2)

or $G(t) = 0$ in (3)

we will first try to solve homogeneous equations

(2) $P(t)y'' + Q(t)y' + R(t)y = 0$

later we will show that one can solve inhomogeneous problems based on the solution of the homogeneous problem

We start w/ constant coefficients

(4) $ay'' + by' + cy = 0$

this is significantly easier than (2) which is hard to do in general (with explicit solutions anyway)

<u>Example:</u> $\begin{cases} y'' - y = 0 \\ y(0) = 2, \quad y'(0) = -1 \end{cases}$

$y'' = y$ familiar?

$y_1(t) = e^{t}$, $y_2(t) = e^{-t}$ come to mind

let's try to build a ~~to~~ solution of the IVP based on these. Since the equation is <u>linear</u>

$$y(t) = \underbrace{c_1 y_1(t) + c_2 y_2(t)}_{\text{linear combination / super-position}} \quad \text{solves as well}$$

$2 = y(0) = c_1 e^{0} + c_2 e^{-0} = c_1 + c_2$

$-1 = y'(0) = c_1 e^{0} - c_2 e^{-0} = c_1 - c_2$

this is a 2x2 linear system

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{-1+-1}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}$$

so $y(t) = \frac{1}{2}e^{t} + \frac{3}{2}e^{t}$ solves the IVP.

~~In general~~

For the more general equation (4) we get the idea to try exponential type solutions

$$\begin{cases} y(t) = e^{rt} \\ y'(t) = re^{rt} = ry \\ y''(t) = r^2 y \end{cases}$$

, plugging in to (4)

we get

$$(ar^2 + br + c)y = 0$$

since $y = e^{rt} \neq 0$ it must hold that

(5) $\underline{ar^2 + br + c = 0}$

(5) is called the characteristic equation

for the ODE (4).

Since (5) is quadratic equation there

are two roots which may be

(i) real and different

(ii) real and repeated

(iii) complex conjugates.

start w/ case (i) (which was the

case of our example).

If there are two real roots $r_1 \neq r_2$

then $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ both

solve (4)

and thus

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \quad \text{solves as well.}$$

for any pair of c_1, c_2 constants.

to solve an IVP

$$\begin{cases} y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$$

we plug in

$$y(t) = c_1 e^{r_1(t-t_0)} + c_2 e^{r_2(t-t_0)}$$

$$y_0 = y(t_0) = \cancel{e^{r_1 t_0}} c_1 + \cancel{e^{r_2 t_0}} c_2$$

$$y_0' = y'(t_0) = r_1 \cancel{e^{r_1 t_0}} c_1 + r_2 \cancel{e^{r_2 t_0}} c_2$$

system of two linear equations in two unknowns for c_1, c_2.

since $r_1 \neq r_2$ it can be solved

$$\cancel{e^{r_1 t_0}} \underbrace{\begin{pmatrix} 1 & 1 \\ r_1 & r_2 \end{pmatrix}}_{} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

A matrix is invertible since determinant

$$\det(A) = r_2 - r_1 \neq 0$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{r_2 - r_1} \begin{pmatrix} r_2 & 1 \\ -r_1 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

$$c_1 = \frac{1}{r_2 - r_1} (r_2 y_0 + y_0')$$

$$c_2 = \frac{1}{r_2 - r_1} (-r_1 y_0 + y_0')$$

Example $y'' + 5y' + 6y = 0$ find general solution

plug in $y(t) = e^{rt}$

$$(r^2 + 5r + 6) e^{rt} = 0$$

so need to solve $r^2 + 5r + 6 = 0$

~~Next~~ factor to $(r+2)(r+3) = 0$

$\cancel{(r+5)(} \quad \cancel{(r-6)(r-1)}$

$\cancel{(r+6)(r+1)}$

two real distinct roots $-2, -3$.

$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$ is the general

form of solution.

w/ initial data $y(0) = 2$, $y'(0) = 3$

$\longrightarrow$

$2 = y(0) = c_1 + c_2$

$3 = y'(0) = -2c_1 - 3c_2$

solving $\quad c_1 = 2 - c_2$

$3 = -2(2 - c_2) - 3c_2$

$= -4 + 2c_2 - 3c_2 = -4 - c_2$

$c_2 = -4 - 3 = -7$

$c_1 = 2 - c_2 = 9$

so $\quad y(t) = 9e^{-2t} - 7e^{-3t}$ solves IVP

plot $\quad y(t) \to 0$ as $t \to \infty$

~~y(t) →~~ as $t \to -\infty$

$$\frac{e^{-2t}}{e^{-3t}} = e^{t} \to 0 \text{ as } t \to -\infty$$
$$\to \infty \text{ as } t \to \infty$$

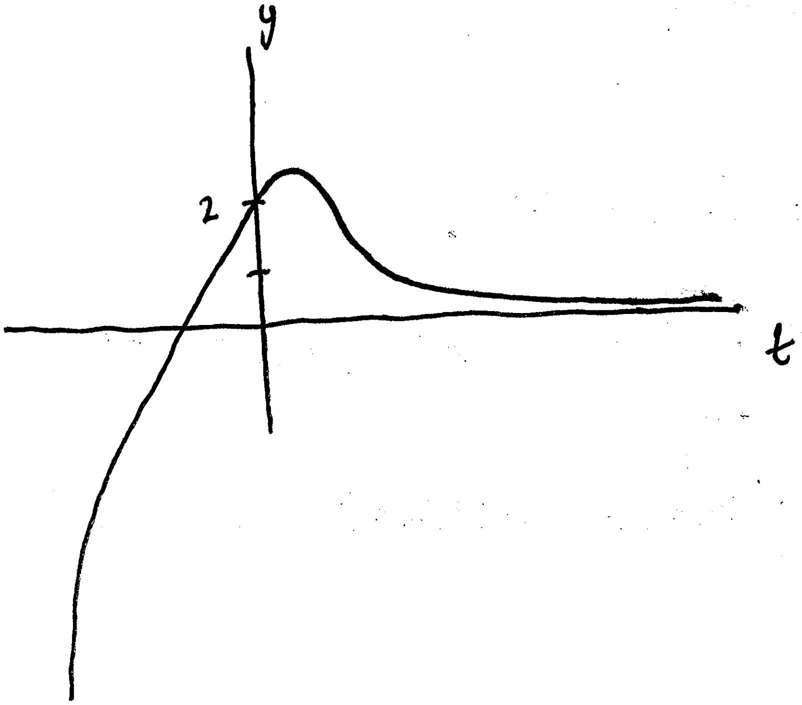
so e^{-3t} much bigger ~~so~~ for $t \to -\infty$

$y(t) \to -\infty$ like $-7e^{-3t}$

e^{-2t} much bigger for $t \to \infty$

~~so~~ ~~y(t)→~~

also we know $y(0)=2$, $y'(0)=3$



3.2 Solutions of Linear Homogeneous Eqns, the Wronskian

~~We return to~~ $y''+p(t)y'+q(t)y=0$

Let p, q be continuous functions on an open interval I, $\alpha<t<\beta$ (possibly $\alpha=-\infty$ and/or $\beta=+\infty$)

for a function ϕ which is twice differentiable on I we define the differential operator L by

$$L[\phi](t)=\phi''(t)+p(t)\phi'(t)+q(t)\phi(t).$$

i.e. $L[\phi]$ is a new function of t on I

e.g. if ~~$\phi(t) = t^2$~~

~~$L[\phi](t) = 2 + p(t)(2t) + q(t)\cdot t^2$~~

$p(t) = t^2$ $\quad q(t) = (1+t)$ and $\phi(t) = \sin t$

$$L[\phi](t) = -\sin t + t^2 \cos t + (1+t)\sin t$$

We can also write

$L = D^2 + pD + q$ where D is the derivative operator
$D[\phi] = \phi'$

In this notation the ODE becomes

$$(1) \begin{cases} L[\phi](t) = 0 & \text{for } t \in I. \\ \text{w/ IC } & y(t_0) = y_0, \; y'(t_0) = y_0' \end{cases}$$

Theorem 1 $\begin{cases} L[y](t) = g(t) \\ y(t_0) = y_0 \;,\; y'(t_0) = y_0' \end{cases}$

where p, q, g continuous on open interval I containing t_0. Then there is a unique solution $y = \phi(t)$ of this IVP and the solution exists on the entire time interval I.

Now suppose that y_1, y_2 are solutions of

$$L[y_1] = 0 \quad , \quad L[y_2] = 0$$

Theorem 2 Any linear combination $c_1 y_1 + c_2 y_2$ solves

$$L[c_1 y_1 + c_2 y_2] = 0 \quad .$$

proof:

$$L[c_1 y_1 + c_2 y_2] = [c_1 y_1 + c_2 y_2]'' + p[c_1 y_1 + c_2 y_2]' + q[c_1 y_1 + c_2 y_2]$$

$$= c_1 y_1'' + c_2 y_2'' + p c_1 y_1' + p c_2 y_2' + q c_1 y_1 + q c_2 y_2$$

$$= c_1 [y_1'' + p y_1' + q y_1] + c_2 [y_2'' + p y_2' + q y_2]$$

$$= c_1 L[y_1] + c_2 L[y_2] = 0 + 0 = 0$$

Thus $c_1 y_1 + c_2 y_2$ is a solution as well.

This is called the <u>principle of super-position</u>

Now we want to know whether this family $c_1 y_1 + c_2 y_2$ encompasses all possible solutions or are there solutions of other forms?.

If we can choose c_1, c_2 to satisfy any initial value problem then we would know from the uniqueness theorem that we have found all possible solutions.

we want to solve

$$y(t_0) = y_0$$
$$y'(t_0) = y_0'$$

which means

$$c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$$
$$c_2 y_1'(t_0) + c_2 y_2'(t_0) = y_0'$$

which is a linear system

(2) $$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

this can be solved uniquely for every (y_0, y_0') if and only if

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0) \neq 0$$

If $W \neq 0$ then (2) has a unique solution

If $W = 0$ then (2) ~~[illegible]~~ cannot be solved for all y_0, y_0'

W is called the <u>Wronskian Determinant</u> or the <u>Wronskian</u>.

<u>Theorem</u> 3 Suppose y_1, y_2 solve

$$L[y] = 0$$

and IC $y(t_0) = y_0$, $y'(t_0) = y_0'$ are assigned. Then the IVP can be solved for all y_0, y_0' if and only if ~~$W(t_0) \neq 0$~~.

$$\boxed{W = y_1(t_0)\, y_2'(t_0) - y_2(t_0)\, y_1'(t_0) \neq 0}$$

<u>Example</u> $y'' + 5y' + 6y = 0$

We found $y_1(t) = e^{-2t}$, $y_2(t) = e^{-3t}$

the Wronskian at t_0 is

$$W = e^{-2t_0} \cdot (-3e^{-3t_0}) - e^{-3t_0}(-2e^{-2t_0})$$

$$= 2e^{-5t_0} - 3e^{-5t_0} = -e^{-5t_0} \neq 0 \text{ ever}$$

Theorem 4 Suppose that y_1 and y_2 solve

$$L[y] = 0 \qquad (*)$$

Then the family

$$y = c_1 y_1 + c_2 y_2$$

w/ arbitrary c_1, c_2 includes every solution of $(*)$ ~~iff~~ if and only if there is a point t_0 where $W(t_0) \neq 0$.

proof: Let ϕ be a solution of

$$L[\phi] = 0$$ on $\mathbb{R}$, let t_0 s.t. $W(t_0) \neq 0$

Call $y_0 = \phi(t_0)$

$y_0' = \phi'(t_0)$

since $W(t_0) \neq 0$ $\exists$ c_1, c_2 with

$y(t) = c_1 y_1 + c_2 y_2$ solves the IVP

$$\begin{cases} y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases} \quad L[y] = 0$$

So by uniqueness theorem $y(t) = \phi(t)$ for all $t \in \mathbb{R}$

On the other hand if $W = 0$ for all t_0 then there are values of y_0, y_0' so that the linear system (2) has no solution c_1, c_2.

Choose $\phi(t)$ solving IVP

$$\begin{cases} L[\phi] = 0 \\ \phi(t_0) = y_0, \quad \phi'(t_0) = y_0' \end{cases}$$

for such a pair y_0, y_0'

such ϕ exists from existence and uniqueness theorem, but it cannot be written as

$$\phi = c_1 y_1 + c_2 y_2$$

<u>Note</u>: One confusing point here is that for both directions we seemed to only use <u>one</u> point to where $W(t_0) \neq 0$ resp $W(t_0) = 0$.

I claim that if $W(t_0) = 0$ at some point then $W(t) \equiv 0$.

Theorem 5 (Abel's theorem)

If y_1, y_2 solve
$L[y] = 0$ then
$$W(y_1, y_2)(t) = c\exp\left[-\int p(t)\,dt\right]$$
where c depends on y_1, y_2 but not t.
thus $W(y_1, y_2)(t) \equiv 0$ or is never 0.

proof $W = y_1 y_2' - y_2 y_1'$
$$W' = y_1 y_2'' + y_1' y_2' - y_2' y_1' - y_2 y_1''$$
$$= y_1 y_2'' - y_2 y_1'' = y_1(-p(t)y_2' - q(t)y_2)$$
$$- y_2(-p(t)y_1' - q(t)y_1)$$
$$= -p(t)(y_1 y_2' - y_2 y_1') - q(t)(y_1 y_2 - y_1 y_2)$$

$= -p(t) W \neq 0$

so $\begin{cases} W' = -p(t) W \\ W(t_0) = y_1(t_0) y_2'(t_0) - y_2(t_0) y_1'(t_0) \end{cases}$

which has a unique solution given in the theorem.

If the Wronskian $W(y_1, y_2) \neq 0$ then we say that y_1, y_2 form a <u>fundamental set of solutions</u>

and $y = c_1 y_1 + c_2 y_2$ is a <u>general solution</u>

<u>Theorem 6</u> Consider (2)

$$L[y] = y'' + p(t) y' + q(t) y = 0$$

w/ p, q cts on I, choose some $t_0 \in I$, then let y_1 solve

(2) w/ $y(t_0) = 1$, $y'(t_0) = 0$

and y_2 solve

(2) w/ $y(t_0) = 0$, $y'(t_0) = 1$

then y_1, y_2 is fundamental solution

proof

$$W[y_1, y_2](t_0) = y_1(t_0)\, y_2'(t_0) - y_1'(t_0)\, y_2(t_0)$$
$$= 1 \cdot 1 - 0 \cdot 0 = 1 \neq 0$$

(the "hard" part is the existence of y_1, y_2 from Theorem 1)

Finally we make note of a useful ~~thm~~ result for later

Thm If $y(t) = u(t) + i v(t)$ u, v real valued solves $L[y] = 0$ then u, v solve

$$L[u] = 0, \quad L[v] = 0.$$

(since p, q are real valued)