Fisher ~~Chapter 4~~ : Functions of a complex variable

1. The Complex Plane

A complex number

$z = x + iy$ where $x, y \in \mathbb{R}$ (real numbers)

and ~~$i = \sqrt{-1}$~~ i satisfies

$(i)^2 = (i) \cdot (i) = -1$

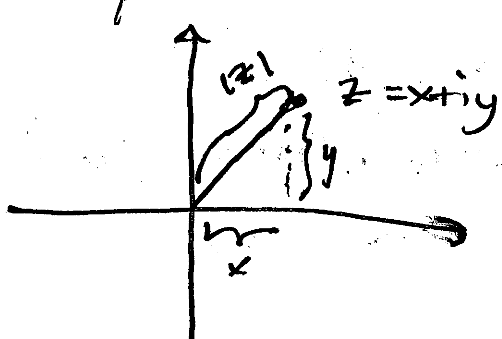
$x = \operatorname{Re} z$ the real part of z

$y = \operatorname{Im} z$ the imaginary part of z

The modulus / absolute value / magnitude of z is

$|z| = \sqrt{x^2 + y^2}$ when $z = x + iy$

each complex number z corresponds to a point in the x,y-plane



this is called $\mathbb{C}$ the complex plane like we visualized real numbers as lying on the real line, $\mathbb{R}$, we view complex numbers as coordinates in the plane $\mathbb{C}$.

you may recognize

$|z|$ as the Euclidean distance from (x,y) to $(0,0)$ in $\mathbb{R}^2$.

Some basic inequalities, $z = x+iy$, involving $\operatorname{Re} z$, $\operatorname{Im} z$

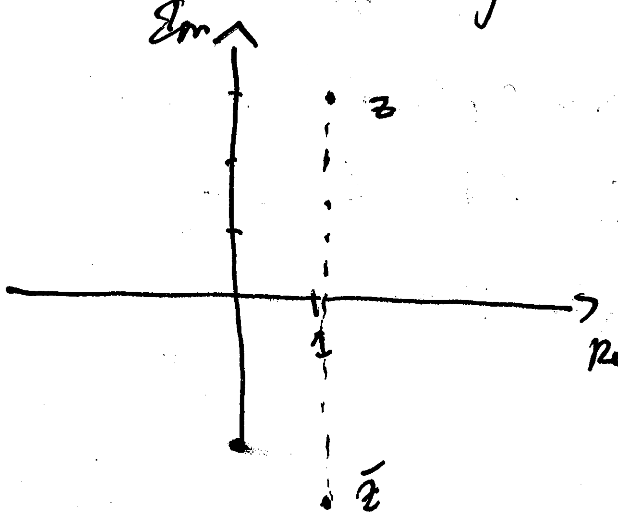
$$|x| \le |z| \quad , \quad |y| \le |z|$$

and $$|z| \le |x| + |y|$$

another important concept is the complex conjugate of z

$$\bar{z} = x - iy$$

~~note $z\bar{z} =$~~



e.g. $z = 1 + 3i$

$\bar{z} = 1 - 3i$

reflect through the real axis

basic algebra

$z = x+iy \qquad w = s+it$

$z+w =$

$zw = \cancel{xy}\ (xs - yt) + i(ys + tx)$

$\cancel{\frac{1}{z} = \bar{z}}$

note $\qquad z\bar{z} = x^2 + y^2 + i(0)$

$\qquad\qquad = |z|^2$

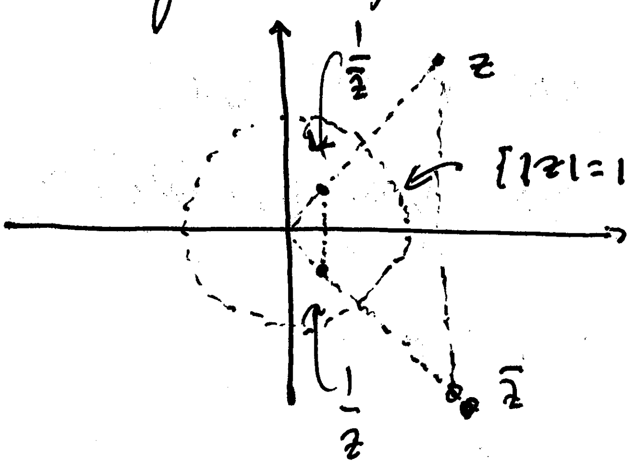
(this is the main reason for defining the complex conjugate fyi)

from useful algebra

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$$

geometrically $\qquad$ note $\left|\frac{1}{z}\right| = \frac{|\bar{z}|}{|z|^2} = \frac{|z|}{|z|^2}$

$\qquad = \frac{1}{|z|}$



we used also

$$|\bar{z}| = |z|$$

another useful fact

$$|zw| = |z||w|$$

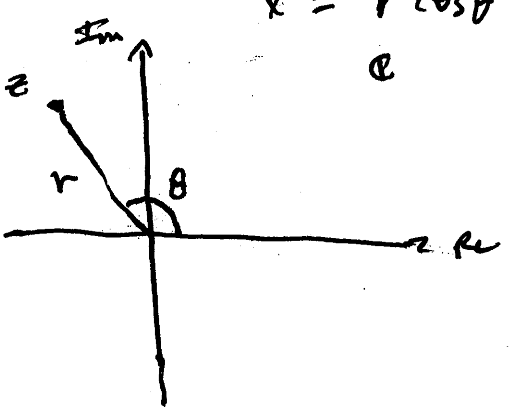
$$\left(\begin{aligned} |zw|^2 &= (xs-yt)^2 + (xt+ys)^2 \\ &= x^2s^2 + y^2t^2 + x^2t^2 + y^2s^2 \\ &= (x^2+y^2)(t^2+s^2) \\ &= |z|^2|w|^2 \end{aligned} \right)$$

$$\overline{zw} = \bar{z}\,\bar{w} \quad :$$

$$\begin{aligned} \overline{zw} &= xs - yt - i(xt+ys) \\ &= (x-iy)(s-it) = \bar{z}\,\bar{w} \end{aligned}$$

An extremely important way of understanding the complex plane is the polar representation

$x = r\cos\theta \qquad y = r\sin\theta$



$r = |z|$

so

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

we already recognize $r = |z|$ as the modulus of the complex number

θ is called $\arg z$ the argument of the complex number.

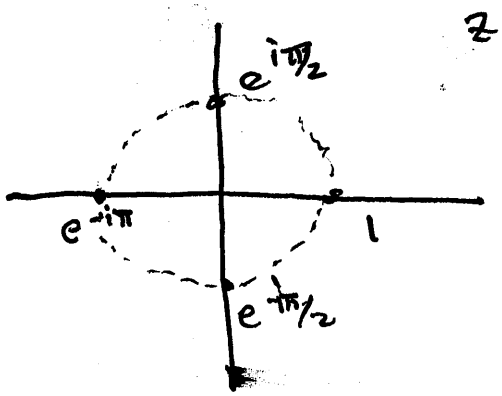
Actually there is some ambiguity here since $\arg z$ is not uniquely define:

For <u>any</u> θ so that $z = re^{i\theta}$ we write $r = \arg \theta$

When we make a concrete choice we call $\operatorname{Arg} z$ to be the <u>unique</u> $\theta_0 \in [-\pi, \pi)$ so that

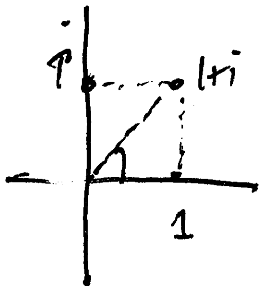
$$z = re^{i\theta}$$

~~e.g.~~



Some examples

$z = 1+i$ $\qquad |z| = (1+1)^{1/2} = \sqrt{2}$



$\operatorname{Arg} z = \frac{\pi}{4}$

$z = \sqrt{2}\, e^{i\frac{\pi}{4}}$

$z = i$ $\qquad |z| = 1$

$\operatorname{Arg} z = \frac{\pi}{2}$

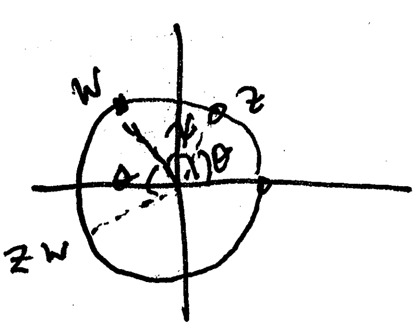
$i \cancel{z} = e^{i\frac{\pi}{2}}$

more geometry; $\quad z, w, \quad \arg z = \theta$
$\arg w = \psi$

$$zw = |z| e^{i\theta} |w| e^{i\psi}$$
$$= |z||w| e^{i(\theta+\psi)}$$

e.g. if $|z| = |w| = 1$ $\quad z, w$ in unit circle

$zw = e^{i(\theta+\psi)}$ $\qquad$ angles add



So complex multiplication zw

stretches the magnitude

by $|z|\,|w|$

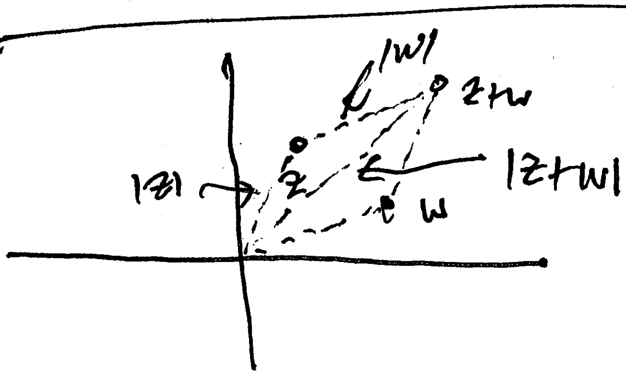
and rotates by adding the

arguments $\arg z + \arg w$

(1.2) More Geometry of the Complex Plane

$$
\begin{aligned}
|z+w|^2 &= (x+s)^2 + (y+t)^2 \\
&= x^2+s^2+y^2+t^2 + 2(xs+yt) \\
&= |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\bar{w}) \\
&\le |z|^2 + |w|^2 + 2|z\bar{w}| \\
&= |z|^2 + |w|^2 + 2|z||w| \\
&= (|z|+|w|)^2
\end{aligned}
$$

$\boxed{|z+w| \le |z| + |w| \quad \text{The triangle inequality}}$



(like adding vectors

also by triangle inequality

$$|z| \leq |z-w| + |w|$$

$$|z-w| \geq |z| - |w|$$

same reasoning

$$|z-w| \geq |w| - |z|$$

so

$\boxed{|z-w| \geq \big||w| - |z|\big| \quad \text{reverse triangle inequality}}$

Equations of lines in the plane

~~$y = mx+b$ usual equation for non-vertical lines in the plane~~
~~m, b real~~

$Re(az+b) = 0$ is a line in the plane

$a, b \in \mathbb{C}$

also $\operatorname{Im}(az+b)=0$

(since ~~Re Im~~

$\operatorname{Re}(iz) = \operatorname{Re}(ix - y)$

$= -\operatorname{Im}(z)$)

So $\operatorname{Im}(az+b) = 0$

if and only if

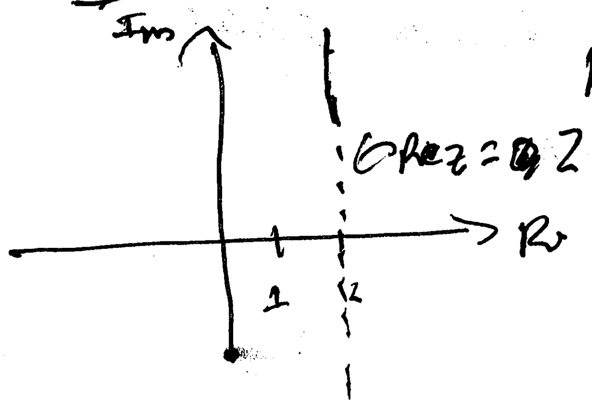
$-\operatorname{Re}(\underbrace{ia}_{\tilde{a}}z + \underbrace{ib}_{\tilde{b}}) = 0$

$0 = \operatorname{Re}(\tilde{a}z+\tilde{b}) = Ax - By + \operatorname{Re}(\tilde{b})$

$\tilde{a} = A + iB$ more recognizable equation of a line

E.g. $\operatorname{Im} z = 0$ → real axis

$\operatorname{Re} z = 0$ → imaginary axis

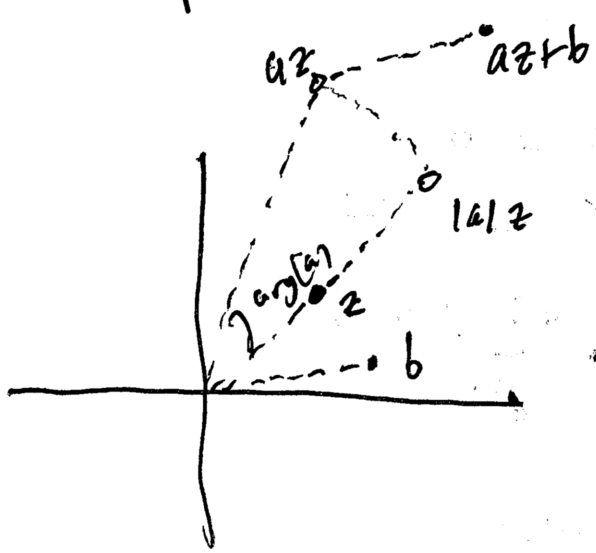


$\operatorname{Re} z = 2$

Ex $f(z) = az + b$ is a transformation of $\mathbb{C}$

lets suppose ~~a = 1~~ ~~$|a| =$~~

decompose $az = |a| e^{i \arg a} |z| e^{i \arg z}$



scale rotate

$f(z) = az + b$

scale and rotate, shift

Roots, of complex numbers (square root, nth roots etc)

actually the whole reason for the subject of complex variables started with trying to solve

$$x^2 + 1 = 0$$

i.e. to find $(-1)^{1/2}$

Consider $z = |z| e^{i\theta}$

we want to solve for w so that

$w^n = z$, $w = |w| e^{i\psi}$

~~$|w|^n =$~~

$w^n = |w|^n e^{in\psi} = |z| e^{i\theta} = z$

~~by~~ must have $|w| = |z|^{\frac{1}{n}}$

(well defined since $|z|$ positive real)

$n\psi = \theta$

but also

$n\psi = \theta + 2\pi k$ works as well

since $e^{in\psi} = e^{i\theta} e^{i2\pi k} = e^{i\theta}$

$\psi = \frac{\theta}{n} + \frac{2\pi k}{n}$

how many of these lead to distinct complex numbers?

$k = 0, 1, \ldots, n-1$ after that

$$w_{n+k} = |w|\, e^{i\frac{\theta}{n} + i\frac{(n+k)}{n}2\pi} = |w|\, e^{\frac{i\theta}{n} + \frac{ik}{n}2\pi} = w_k$$

similarly for w_{ln+k} l any integer

so there are <u>n distinct roots</u>

an important special case is

the <u>roots of unity</u>

<u>E.g.</u> find the 4th roots of 1

there are 4 distinct roots

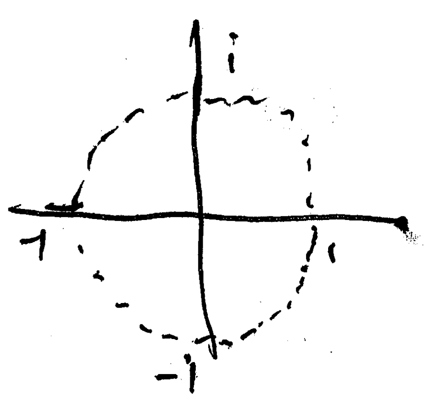
$$1 = 1 \cdot e^{i \cdot 0}$$

$$\cancel{w =}$$

$$w_k = e^{i\frac{k}{4}\cdot 2\pi} = e^{i\frac{k\pi}{2}} \quad \text{for } k = 0, 1, 2, 3$$

$$1,\ e^{i\frac{\pi}{2}},\ e^{i\pi},\ e^{i\frac{3\pi}{2}}$$

$$\qquad " \; i \qquad " \; -1 \qquad " \; -i$$



dividing the unit circle up into fourths

Circles

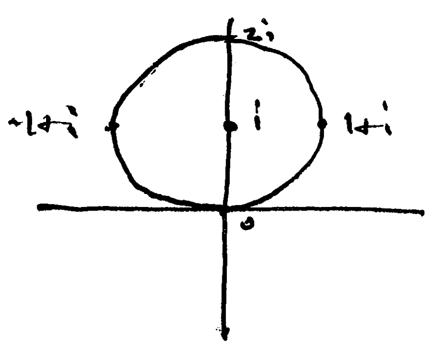
A circle is the set of points equidistant from a given point. If z_0 is the center and r the radius then the circle is described by

$$|z - z_0| = r$$

A special important case is the <u>unit circle</u> called $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$

and the <u>unit disk</u> $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

example $|z - i| = 1$



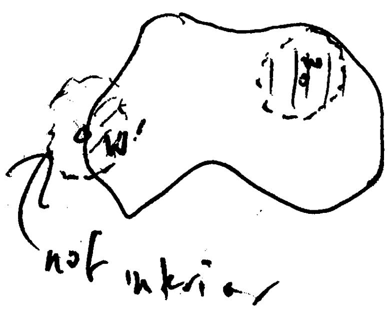
1.3 Subsets of the plane $\boxed{\text{8 Sept}}$

~~This~~ These concepts should be familiar from 200.

~~Def~~

The set of points $|z - z_0| < R$ is called the <u>open disk</u> of radius R centered at z_0. ~~We call this $D(z_0, R)$~~ $D(z_0, R)$.

If ~~D~~ U is a subset of $\mathbb{C}$ and $w \in$ ~~D~~ U we say w is <u>interior</u> if there is $R > 0$ s.t. ~~the open disk of radius R centered at~~ ~~$D(z$~~ $D(w, R) \subseteq U$.

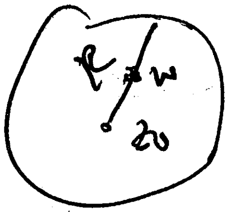


A set U is called <u>open</u> if all its points are <u>interior</u>.

ex open disks $\{z : |z-z_0| < R\}$ are open

if $w \in D(z_0, R)$

let $\varepsilon = \min R - |w - z_0| > 0$



$D(w, \varepsilon) \subseteq D(z_0, R)$

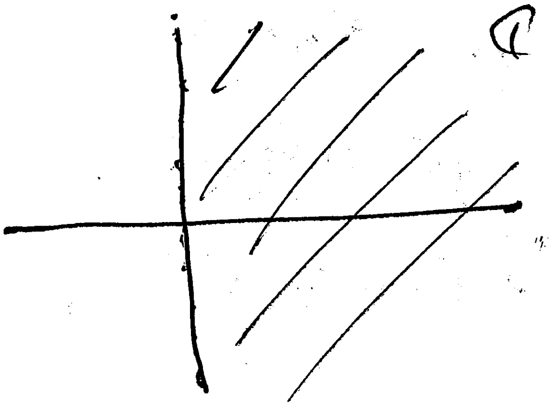
If $z \in D(w,\varepsilon)$

$$|z - z_0| \leq |z - w| + |w - z_0|$$

$$< R - |w - z_0| + |w - z_0|$$

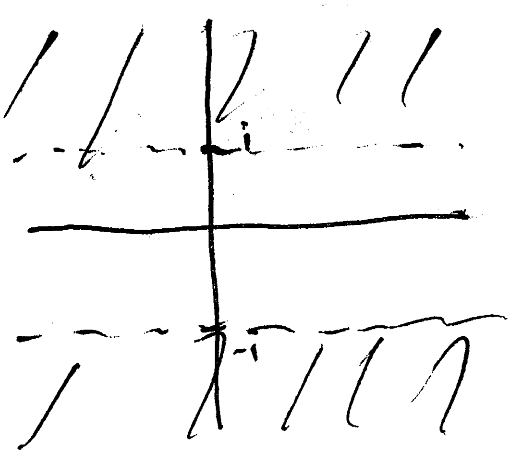
$$= R.$$

ex $\operatorname{Re} z > 0$ is open



open

ex $|\operatorname{Im} z| > 1$



open

A point p is called a <u>boundary point</u> of U if for every $R>0$

$D(p,R) \cap U \neq \phi$ <u>and</u> $D(p,R) \cap U^c \neq \phi$

$\partial U = \{ p \in \mathbb{C} : p \text{ is a bdry point of } U \}$

<u>ex</u> $\partial D(z_0,R) = \{ |z-z_0| = R \}$

<u>ex</u> $\partial \{ \operatorname{Re} z > 0 \} = \{ \operatorname{Re} z = 0 \}$

etc.

A set E is called <u>closed</u> if

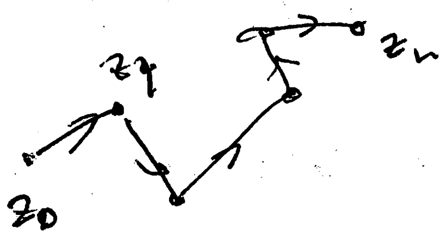
$\mathbb{C} \setminus E$ is open.

boundary points of E are boundary points of E^c and so E is closed iff it contains all its bounding points

U is open iff it contains none of its bdry points.

Connected Sets

A polygonal curve is a finite union of directed line segments $P_1, P_2, \ldots P_n$



i.e. it has a piecewise linear parametrisation

$$\gamma(t) = \{ \cancel{t z_j +}$$

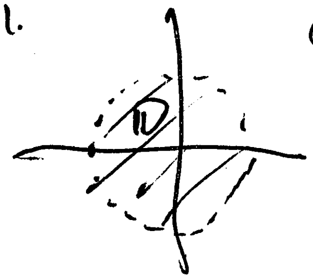
$$[1-(t-t_j)]\, z_j + (t-t_j) z_{j+1} \quad \text{for } t_j \le t \le t_{j+1}$$

$$t_j = t_{j-1} + 1 \quad \text{i.e. } t_j = j$$

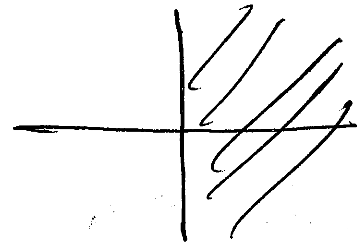
~~and~~ An open set U is connected if each pair $p, q \in U$ may be joined by a polygonal curve in U.

Ex

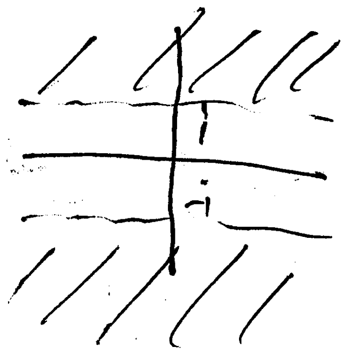
1. $\mathbb{C}$



2. $\mathrm{Re}\, z > 0$



3. $|\mathrm{Im}\, z| > 1$



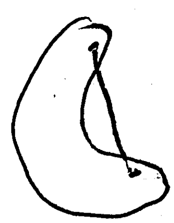
not connected.

An open connected set is called a <u>domain</u>

A set E is convex if the line segment pq joining each pair $p, q \in E$ is contained in E. Convex open sets are domains

e.g.

convex



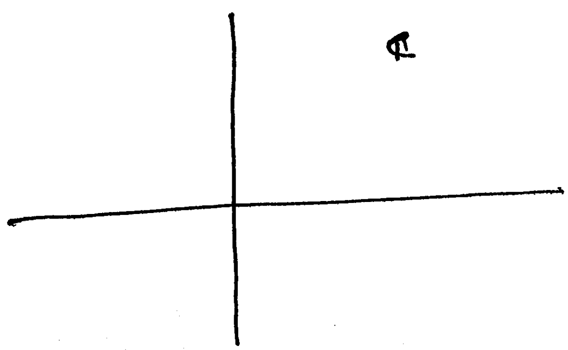
not convex

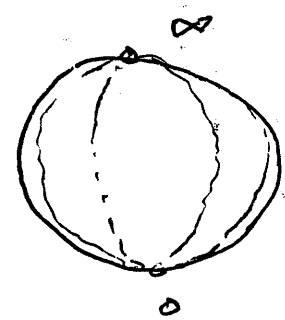
The point at infinity

~~We $\mathbb{C}$~~

~~A useful conceptual~~

It can be useful conceptually to think of ∞ as another point





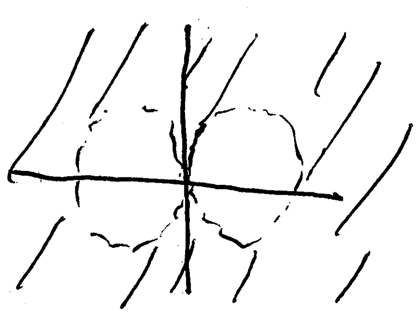
$\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$

lines in $\mathbb{C}$ are circles through ∞ in $\mathbb{C}^*$.

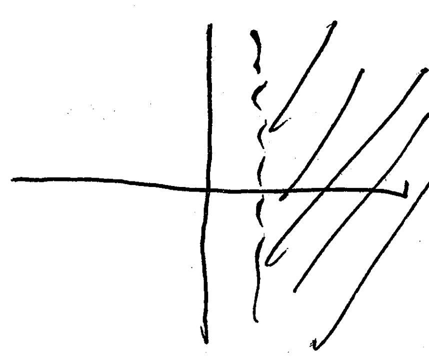
We say ∞ is an interior point of U if $\exists\ M > 0$ so that

$\{|z| > M\} \subseteq U$.

e.g.

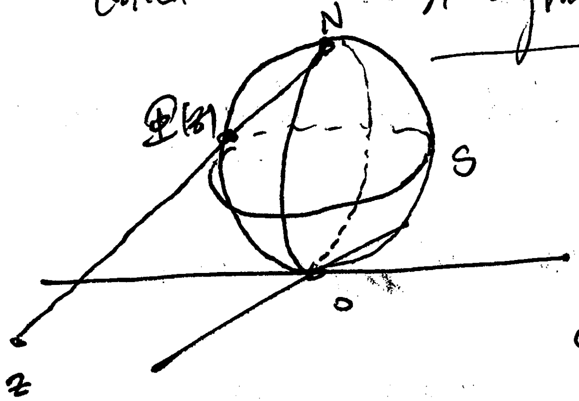


∞ is interior



∞ is bdry point

There is a way to make this identification of $\mathbb{C}^*$ w/ the sphere more rigorous called stereographic projection



Draw line from north ~~pl~~ pole N to $z \in \mathbb{C}$. & and ~~and~~ identify z w/ the unique point on S where the line passes through. Define a invertible continuous mapping $\Phi : \mathbb{C} \to S \setminus \{N\}$

~~Open sets~~

∞ being an interior point of U in $\mathbb{C}_*$ corresponds to the usual notion of ∞ being an interior point of $\Phi(U)$ in the sphere S (if we use disks in the sphere coming from geodesic distance)

1.4 functions and limits

A function f of a complex variable z.

is a rule that assigns to each $z \in D$,

D some specified set, called the domain of definition

a value $f(z) \in \mathbb{C}$. The collection of all

possible values of $f(z)$, $z \in D$, is

called the range of f.

ex $f(z) = 4z^2 + z + 1$ w/ domain all of $\mathbb{C}$

what is range?

$W = f(z) = 4z^2 + z + 1$ this is just

a root of a quadratic, for every

w there ~~are~~ is always at least 1 values of z

so that $f(z) = w$.

example $f(z) = \frac{1}{z}$ has domain $\mathbb{C} \setminus \{0\}$

Its range is $\mathbb{C} \setminus \{0\}$ as well

since ~~$\frac{1}{z} \neq 0$ then~~ $\frac{1}{z} = \frac{\bar{z}}{|z|^2} \neq 0$ ever for $z \in \mathbb{C}$

but to solve for $w = \frac{1}{z}$

take $z = \frac{1}{w}$.

If I restricted the domain to just the $\mathbb{D} \setminus \{0\}$ what would be the range?

Limits

$\{z_n\}_{n=1}^{\infty}$ a sequence of complex numbers,

we say that z_n has limit A

or converges to A if

and write

$\lim_{n \to \infty} z_n = A$ or $z_n \to A$

if for every $\varepsilon > 0$ there exists N suff large

s.t. $\forall$ $n \geq N$

$|z_n - A| \leq \varepsilon$.

e.g. 1. $z_n = 1 + \frac{i}{n}$

2. $z_n = \left(-\frac{1}{2}\right)^n + i\left(1 - \frac{2}{n}\right)$

Simple facts

if $z_n \to A$

then $|z_n| \to |A|$

and if $z_n \to A$, $w_n \to B$

and $\lambda \in \mathbb{C}$ fixed

$$z_n + \lambda w_n \to A + \lambda B$$

$$z_n w_n \to AB$$

Then are just special cases of

~~lim~~ continuous functions

Consider

~~We say~~ $f: S \subseteq \mathbb{C} \to \mathbb{C}$ ~~is continuous~~

~~at z_0 if~~

We want to consider the limiting value of f as z approaches some point z_0

We say f has limit L at z_0 and write

$$\lim_{z \to z_0} f(z) = L \quad \text{or} \quad f(z) \to L \text{ as } z \to z_0$$

if for all $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$|z - z_0| \leq \delta \Rightarrow |f(z) - L| \leq \varepsilon.$$
not $z_0 \in S$

Can define limit at infinity similarly

We say f is cts at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

e.g. 1. $f(z) = |z|^2$

2. $f(z) = \frac{1}{1-z}$

Infinite Series

how to interpret $\sum_{n=1}^{\infty} z_n$?

$S_n =$

look at partial sums

$$S_n = \sum_{j=1}^{n} z_j \qquad n = 1, 2, \ldots, \infty$$

if the sequence S_n converges

then we define the infinite sum

as the limit of the partial sums.

When $\sum_{j=1}^{\infty} |z_j|$ converges we

say that the sum is absolutely convergent

this implies that the complex

partial sums converge as well

the reason is that for $m \le n$

$$|S_n - S_m| = \left| \sum_{j=1}^{n} z_j - \sum_{j=1}^{m} z_j \right| = \left| \sum_{j=m}^{n} z_j \right| \le \sum_{j=m}^{n} |z_j|$$

$$\le \sum_{j=m}^{\infty} |z_j| \to 0 \quad \text{as} \quad m \to \infty$$

since we assumed that $\sum_{j=1}^{\infty} |z_j|$ converges

so S_n is a Cauchy sequence

and (for us ~~anyway~~) without proof

Cauchy sequences converge.

~~also~~

An important series is the geometric series

$$\sum_{j=1}^{n} \alpha^j = 1 + \alpha + \alpha^2 + \cdots + \alpha^n \qquad \alpha \in \mathbb{C}$$

$$S_n = \sum_{j=0}^{n} \alpha_j = \frac{1-\alpha^{n+1}}{1-\alpha} \qquad \text{how to prove?}$$

$$S_n + \alpha^{n+1} = S_{n+1} = 1 + \alpha + \cdots + \alpha^{n+1} = 1 + \alpha(1 + \cdots + \alpha^n)$$

$$= 1 + \alpha S_n$$

solve for S_n $\qquad (1-\alpha) S_n = 1 - \alpha^{n+1}$

$$S_n = \frac{1-\alpha^{n+1}}{1-\alpha}$$

Can also see

$$\sum_{j=0}^{\infty} \alpha^j = \lim_{n\to\infty} S_n = \frac{1}{1-\alpha}$$

as long as $|\alpha| < 1$

if $|\alpha| > 1$ $\quad |S_n| \to \infty$

since that $|\alpha^{n+1}| = |\alpha|^{n+1} \to \infty$

if $|\alpha| = 1$ $\quad \frac{1-\alpha^{n+1}}{1-\alpha}$ typically will not

converge (e.g. $\alpha = -1$)
but not clear

example

ratio test if $x_j \geq 0$

$$\alpha = \lim_{j\to\infty} \frac{x_{j+1}}{x_j} \text{ exists}$$

then $\sum_{j=1}^{\infty} x_j$ converges if $\alpha < 1$

" diverges if $\alpha > 1$

proof If $\alpha < 1$

$\exists$ N suff large so that

$$\frac{x_{j+1}}{x_j} \leq \alpha + \frac{1-\alpha}{2} < 1 \quad \text{for all } j \geq N$$

$$\Rightarrow x_j \leq \alpha^{j-N} x_N$$

then

$$\left|\sum_{j=1}^{k} x_j\right| \leq \left|\sum_{j=1}^{N} x_j\right| + \left|\sum_{j=N}^{k} x_j\right|$$

$$\leq N \max_{1\leq j\leq N} |x_j| + x_N \boxed{\sum_{j=0}^{k-N} \alpha^j} \text{ converges}$$

Special functions

$e^z = e^x(\cos y + i\sin y)$ $\qquad z = x+iy$

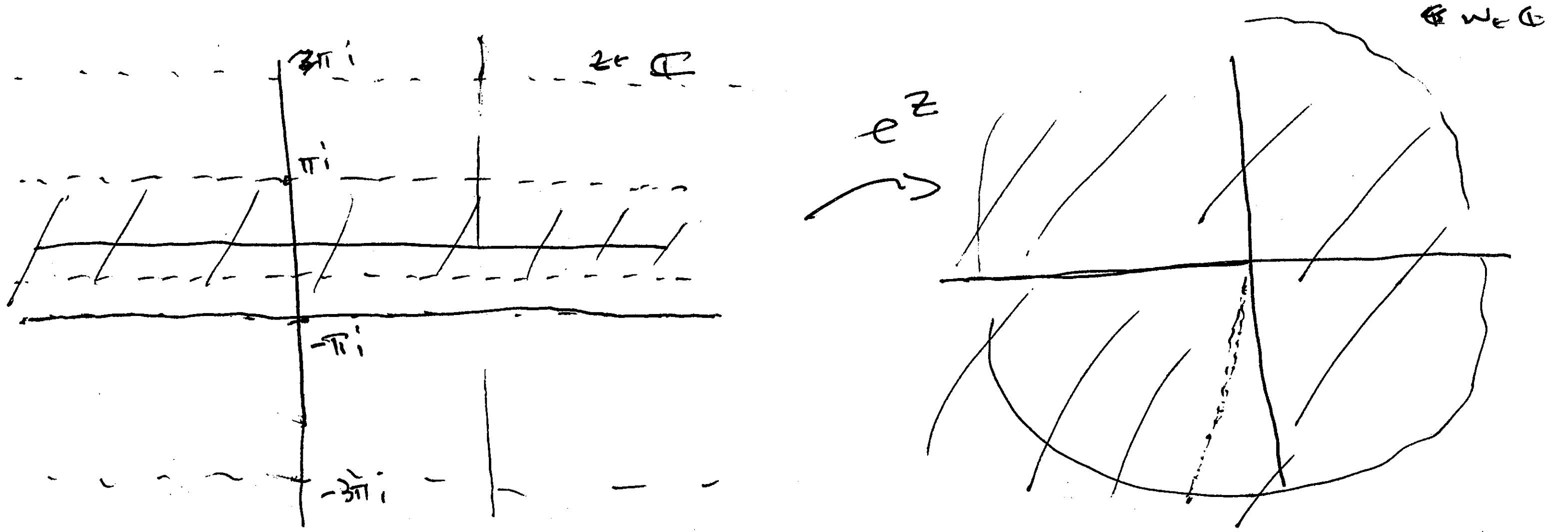
the exponential fn, we've seen before

$|e^z| = e^{\mathrm{Re}\, z}$

$\arg e^z = \mathrm{Im}\, z$

e^z is $2\pi i$ periodic

$e^{z+2\pi k i} = e^z$



horizontal lines $\longrightarrow$ rays from the origin

vertical lines $\longrightarrow$ circles

$(\mathrm{Re}\, z > 0) \rightarrow |w| > 1$

$(\mathrm{Re}\, z < 0) \rightarrow |w| < 1$

note ~~z~~ range of e^z is $\mathbb{C}\setminus\{0\}$

$e^{\operatorname{Re} z}$, $e^{i \operatorname{Im} z}$ both never zero

~~Xx~~ The <u>logarithm</u>

We can also define, with more care, the inverse of exp

We define

$\log z$ to be <u>any</u> $w \in \mathbb{C}$ s.t.

$$e^w = z = re^{i\theta}$$

(note any $w = \log z$ then $w + 2\pi k i = \log z$ also)

we ~~already~~ can see from polar decomposition of z that

$$\log z = \ln|z| + i \underbrace{\arg z}$$

$\mathbb{R}$ natural log of a positive real, uniquely defined

multi-valued argument fun

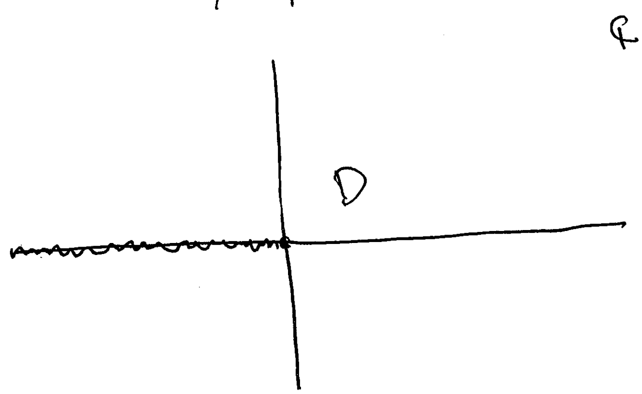
we can also define the principal branch of the log function

$$\mathrm{Log}\, z = \ln|z| + i\,\mathrm{Arg}\, z$$

Choosing a branch of the log function

how can we make $\log z$ into a single valued continuous fcn on a domain D of $\mathbb{C}$?

take, for example $D = \mathbb{C} \setminus (-\infty, 0]$

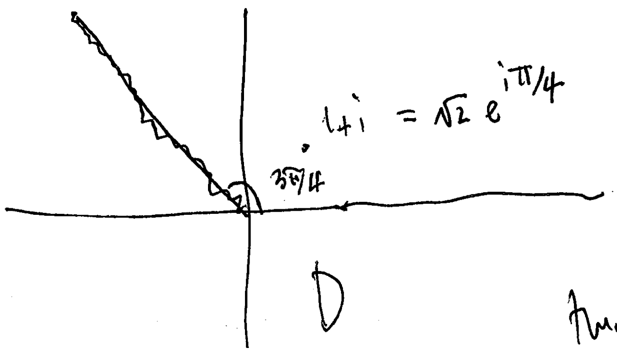


and choose $\arg z = \mathrm{Arg}\, z$ for $z \in D$

then $\mathrm{Log}\, z = \ln|z| + i\,\mathrm{Arg}\, z$ is a cts fcn in D.

It was important that we didn't include some ray from the origin

Could have also done e.g.



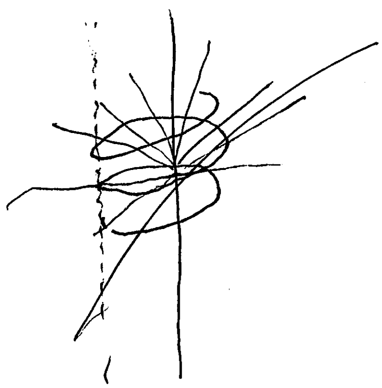
and said

$\arg_0 z \in \left(-\frac{5\pi}{4}, \frac{3\pi}{4}\right)$

then $\log z = \ln|z| + i \arg_0 z$ is cts in D

this is called a branch of log.

There are infinitely many ways to choose a branch. Picture a [illegible] of helix



bad drawing.

~~example~~

Now that we understand log a little better we can also define power functions

$a \in \mathbb{C}$ want to define

$$f(z) = z^a = e^{a \log z}$$

for this we obviously need to specify a branch of Log.

e.g find all the values of

$$(-1)^i = e^{i \log(-1)} = e^{i(\cancel{\ln 1}^{=0} + \pi i + 2\pi k i)}$$

$$= e^{-\pi - 2\pi k} = e^{-(2k+1)\pi} \qquad k = 0, \pm 1, \pm 2, \ldots$$

e.g. solve $z^{1+i} = 4$

$$e^{(1+i)\log z} = 4$$

so $(1+i) \log z = \ln 4 + 2\pi k i$

$$\log z = \frac{1}{1+i}[\ln 4 + 2\pi k i] = \frac{1-i}{2}[\ln 4 + 2\pi k i]$$

$$= \ln 4^{1/2} + \pi k + i[\pi k - \ln 2]$$

so since $\log z = \ln |z| + i \arg z$

$\Rightarrow$ $\ln |z| = \ln 2 + \pi k$ and $\arg z = \pi k - \ln 2$

$|z| = 2e^{\pi k}$ $\qquad e^{i \arg z} = e^{i\pi k - i \ln 2}$

$$= (-1)^k (\cos(\ln 2) - i \sin(\ln 2))$$

We can also define trig fcns

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

Consistent w/ usual cos and sin

when z is real by Euler's formula