MATH 205: Homework 3

Due Wednesday April 18

Problems are from Boller and Sally. I recommend at least thinking about all the exercises, even if they are not assigned, as you read through the textbook. I also recommend trying to prove all the Theorems which are left unproven in the book.

Problem 1. Exercises 5.10.3, 5.10.6, 5.10.7, 5.10.8

Problem 2.. Suppose $E \subset \mathbb{R}^n$ is a bounded set such that, for all $\varepsilon > 0$ there exists an open set $U \supset E$, with ∂U having measure zero, such that $|U| \leq \varepsilon$. Show that E has measure zero.

Problem 3. Let R be a closed bounded rectangle. and $f : R \to \mathbb{R}$ bounded.

(a) Suppose $f : R \to \mathbb{R}$ bounded. Show that if $E = \{x \in R : f \neq 0\}$ has measure zero then f is Riemann integrable on R and,

$$\int_R f\ dx = 0.$$

(b) Suppose that $f, g : R \to \mathbb{R}$ are bounded and integrable. Show that if $D = \{x \in R : f(x) \neq g(x)\}$ has measure zero then,

$$\int_R f\ dx = \int_R g\ dx.$$

(c) Show that if $f : R \to \mathbb{R}$ is Riemann integrable and $\Omega \subset R$ is a bounded set with $\partial\Omega$ having measure zero then,

$$\int_\Omega f\ dx = \int_{\overline{\Omega}} f\ dx = \int_{\Omega^0} f\ dx,$$

where Ω^0 is the interior of Ω.

(d) Show that if Ω, and $V_1, \ldots, V_N \subset \Omega$ are open bounded sets whose boundaries have measure zero, V_j are mutually disjoint, and $\Omega \subset \cup_{j=1}^N \overline{V_j}$ then for any f which is Riemann integrable on Ω,

$$\int_\Omega f = \sum_j \int_{V_j} f.$$

Hint: Show that $\mathbf{1}_\Omega = \sum_j \mathbf{1}_{V_j}$ except for a set of measure zero.

Problem 4. Cylindrical coordinates (r, θ, z) for $\mathbb{R}^3$ are defined by $(x, y, z) = (r\cos\theta, r\sin\theta, z)$. Spherical coordinates were defined in the previous problem 5.10.6. Compute the following integrals by choosing a convenient coordinate system.

(a) $\int_D z^2$ where D is the hemisphere $D = \{x^2 + y^2 + z^2 \leq 1\} \cap \{z \geq 0\}$.

(b) $\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^3 xy\ dzdydx$ (what region is this integral over?)