MATH 275: Homework 8
Due at your Final Exam

Problem 1. The general existence and uniqueness theory for scalar conservation laws of the form,

$$u_t + f(u)_x = 0 \quad \text{in} \quad \mathbb{R} \times (0, \infty) \tag{1}$$

is based on the idea of *entropy solutions*. In this problem we will see the motivation for this notion of solution and show that entropy solutions satisfy the *Lax entropy condition* along shocks. Let η be a smooth convex function, we call this an *entropy*, define the *entropy flux* q by,

$$q(u) := \int_0^u \eta'(v) f'(v) \, dv.$$

The pair (η, q) is called an *entropy pair*.

(a) Show that if u^ε solves the viscous approximation to the conservation law,

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon \quad \text{in} \quad \mathbb{R} \times (0, \infty) \tag{2}$$

and (η, q) is an entropy pair then,

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \leq \varepsilon \eta(u^\varepsilon)_{xx} \quad \text{in} \quad \mathbb{R} \times (0, \infty). \tag{3}$$

Since we expect the physically relevant solutions of (1) to arise as limits of the viscous approximation we expect that (3) should hold in some form for the (correct) solutions of (1). This motivates the definition of an *entropy solution*:

Definition 0.1. We say $u \in L^\infty(\mathbb{R} \times (0, \infty))$ is an *entropy solution* of (1) if:

(i) u is a weak solution of (1)

(ii) for every entropy pair (η, q) and every *non-negative* test function φ compactly supported in $\mathbb{R} \times (0, \infty)$ it holds that,

$$\int_{\mathbb{R} \times (0,\infty)} \eta(u)\varphi_t + q(u)\varphi_x \, dxdt \geq 0. \tag{4}$$

(b) Show that if u is an entropy solution of (1) in a region V of space-time and u is smooth on either side of a smooth parametrized curve (a shock) $C = \{(\gamma(t), t) : t \in I \subset \mathbb{R}\}$ with u, u_t, and u_x uniformly continuous in the regions V_ℓ and V_r to the left and right of C then the shock satisfies the Lax entropy condition,

$$f'(u_\ell(\gamma(t), t)) \geq \gamma'(t) \geq f'(u_r(\gamma(t), t)) \quad \text{for all} \quad t \in I.$$

Here u_ℓ and u_r are the left and right limits of u along C respectively.

Hint: First show that (4) implies a kind of Rankine-Hugoniot condition for $\eta(u)$. Then choose a good entropy/entropy flux pair.

(c) Show that if u is an entropy solution of (1), satisfying that $u(x,t)$ has compact support in x for each $t > 0$, with initial data $u(x, 0) = u_0(x)$ then for every $p \geq 1$,

$$\int_{\mathbb{R}} |u(x,t)|^p \, dx \leq \int_{\mathbb{R}} |u_0(x)|^p \, dx \quad \text{for all} \quad t > 0. \tag{5}$$

Give an example of a *weak solution* of Burger's equation for which (5) does not hold.

Problem 2. Consider the viscous approximation of a scalar conservation law,

$$u_t + f(u)_x = \varepsilon u_{xx} \quad \text{in} \quad \mathbb{R} \times (0, \infty), \tag{6}$$

for a smooth convex flux f. Carefully show that there exists a (non-trivial) travelling wave solution $u(x,t) = v(\frac{x-ct}{\varepsilon})$ connecting two values $u_\ell, u_r \in \mathbb{R}$,

$$v(-\infty) = u_\ell, \; v'(-\infty) = 0 \quad \text{and} \quad v(+\infty) = u_r, \; v'(+\infty) = 0,$$

if and only if $c = (f(u_\ell) - f(u_r))/(u_\ell - u_r)$ and $f'(u_\ell) > c > f'(u_r)$.

Hint: There is some (basic) ODE theory involved in this problem, you can look back at the note I gave you at the beginning of the quarter or come talk to me if you have any issues.

Problem 3. Shearer and Levy: Chapter 13, problem 10.

Remark: Although I am only giving you this one "computational" problem I recommend that you do some of the other problems in Chapter 13 of Shearer and Levy to get some practice computing entropy solutions of scalar conservation laws. For example problems 6 and 9 in Chapter 13, also problem 7 although you would need to read Section 13.2.2 for that.

Note for Problems 4 and 5: The below problems involve the Hopf-Lax formula for the solution of a Hamilton-Jacobi equation,

$$u_t + H(Du) = 0 \ \text{ in } \ \mathbb{R}^n \times (0,\infty) \ \text{ with } \ u(x,0) = g(x),$$

with g Lipschitz continuous and Hamiltonian $H : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$(1)\ H \text{ convex } \ \text{and} \ (2) \lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty.$$

The Lagrangian L is defined as the convex dual (Legendre transform) of H,

$$L(v) = H^*(v) = \sup_{p\in\mathbb{R}^n} \{p \cdot v - H(p)\}.$$

Then the Hopf-Lax formula gives a weak solution of the Hamilton-Jacobi equation by the formula,

$$u(x,t) = \inf_{y\in\mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}.$$

Problem 4. (Evans 2nd Edition, Chapter 3 problem 13) Prove that the Hopf-Lax formula reads

$$\begin{aligned} u(x,t) &= \inf_{y\in\mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\} \\ &= \inf_{y\in B(x,Rt)} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\} \end{aligned}$$

for $R = \sup_{\mathbb{R}^n} |DH(Dg)|$, $H = L^*$. (This proves finite speed of propagation for Hamilton-Jacobi equations with convex H and Lipschitz g.)

Problem 5. (Evans 2nd Edition, Chapter 3 problem 14) Let E be a closed subset of $\mathbb{R}^n$. Show that if the Hopf-Lax formula could be applied to the initial value problem,

$$\begin{cases} u_t + |Du|^2 = 0 & \text{in} \quad \mathbb{R}^n \times (0,\infty) \\ u = \begin{cases} 0 & x \in E \\ +\infty & x \in \mathbb{R}^n \setminus E \end{cases} & \text{on} \quad \mathbb{R}^n \times \{t=0\} \end{cases}$$

it would give the solution,

$$u(x,t) = \frac{1}{4t}\operatorname{dist}(x,E)^2.$$