MATH 275: Homework 6

Due Thursday, May 12

I recommend reading Chapter 6 and 7 (6.2.1 and 6.4-6.5 are less relevant) of Shearer and Levy for some background material on Fourier Series which should complement the lecture material on separation of variables.

Problem 1. Using the method of separation of variables find the unique *bounded* solution of the following problem set in the upper half space $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$:

$$\begin{cases} -\Delta u = 0 & \text{in} \quad x_2 > 0 \\ u(x_1, 0) = g(x_1) \end{cases} \tag{0.1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a smooth 1-periodic function of the form $g(x) = \sum_{n=1}^{N} [A_n \cos(2\pi n x) + B_n \sin(2\pi n x)]$. What is the value of $\lim_{R\to\infty} u(x_1, R)$?

Problem 2. Consider the following heat equation.

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } (0,1) \times (0,\infty) \\ u(0,t) = u(1,t) = 0 & \text{for } t \geq 0 \\ u(x,0) = f(x) & \text{in } x \in [0,1] \end{cases}$$

First let us consider the case when,

$$f(x) = \sum_{n=1}^{N} a_n \sin(n\pi x).$$

(a) Solve for $u(x,t)$ for $t > 0$, can you extend u to be a solution for negative times?

Next let's take

$$f(x) = 1 - |2x - 1| = \sum_{k=1}^{\infty} \frac{8}{\pi^2} \frac{(-1)^k}{(2k+1)^2} \sin((2k+1)\pi x).$$

(b) Solve for $u(x,t)$. [Be careful: Fourier coefficients of f for $\sin(n\pi x)$ is zero, for n even.]

(c) Show that $u(\frac{1}{2}, -s) = +\infty$ for any $s > 0$.

(d) Let $\epsilon \in (0, \frac{1}{2})$. Show that $u(\frac{1}{2} \pm \epsilon, -s)$ is a divergent series, for any $s > 0$. [**Hint**: Try using the largest term in each partial sum.]

Problem 3. [Evans, 2nd Edition, Ch 2 Problem 24] (Equipartition of energy) Let u solve the initial value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in} \quad \mathbb{R} \times (0,\infty) \\ u = \phi, \ u_t = \psi & \text{on} \quad \mathbb{R} \times \{t = 0\}. \end{cases} \tag{0.2}$$

With ϕ and ψ both compactly supported continuous functions. The *kinetic energy* is $k(t) = \frac{1}{2} \int_{\mathbb{R}} u_t(x,t)^2 \, dx$ and the *potential energy* is $p(t) = \frac{1}{2} \int_{\mathbb{R}} u_x(x,t)^2 \, dx$.

(a) Show that $k(t) + p(t)$ is constant in t.

(b) Show that $k(t) = p(t)$ for sufficiently large t.

Problem 4. In one dimension one can prove from D'Alembert's formula that the solution of the wave equation with intial data $(u, u_t) = (\phi, 0)$ satisfies the bound

$$\sup_{x,t} |u(x,t)| \leq \sup_{x} |\phi(x)|. \tag{0.3}$$

Consider the initial value problem for the wave equation in $\mathbb{R}^3$:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in} \quad \mathbb{R}^3 \times (0,\infty) \\ u = \phi, \ u_t = \psi & \text{on} \quad \mathbb{R}^3 \times \{t = 0\}. \end{cases} \tag{0.4}$$

We will show that a uniform in time estimate on the supremum norm (similar to (0.3) in dimension 1) cannot hold for solutions of the wave equation in $\mathbb{R}^3$. Heuristically speaking, initially small (in supremum norm) disturbances can be arranged to concentrate at some future time resulting in very large local oscillations.

(a) Let us first consider a situation with ϕ and ψ smooth and supported in $B(0,1)$. Prove that $u(x,t)$ is supported in the annulus $B(0,t+1)\setminus\overline{B}(0,t-1)$ and,

$$\sup_x |u(x,t)| \le C\frac{1}{t^2}(\sup_x|\phi(x)| + t\sup_x|D\phi(x)| + t\sup_x|\psi(x)|),$$

for some constant C.

(b) Show that if u is a solution of the wave equation for $t\in(0,\infty)$ and $T>0$ then

$$v(x,t) = u(x,T+t) + u(x,T-t) \text{ solves the wave equation in } \mathbb{R}^3\times(0,T)$$

with $v(x,0) = 2u(x,0)$ and $v_t(x,0)=0$.

(c) For every $\varepsilon>0$ small and $M>0$ large give an example of a compactly supported initial data for the wave equation in $\mathbb{R}^3$ so that $\sup_x|\phi|, \sup_x|D\phi|\le\varepsilon$, $\psi(x)=0$ but there is a positive time $T(\varepsilon)$ so that $\sup_x|u(x,T)|\ge M$.

Remark: This problem is an example of a more general principle. For a time reversible equation, if certain initial data leads to decay in time of some norm of the solution, then also there must be initial data which leads to growth in time of that norm.