MATH 275: Homework 5

Due Thursday, May 5

Problem 1. Let $A : \mathbb{R}^n \to S_n$ where S_n is the space of real symmetric $n \times n$ matrices. Suppose that $0 \leq A(x) \leq \Lambda$ in the sense of matrices, i.e. $\Lambda I - A$ and A are non-negative definite. Show that there is at most one smooth solution of the wave type equation,

$$\begin{cases} u_{tt} - \nabla \cdot (A(x)\nabla u) = f(x,t) & \text{in} \quad \mathbb{R}^n \times (0,\infty) \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x) & \text{in} \quad \mathbb{R}^n. \end{cases}$$

Hint: You should show a finite speed of propagation property as we did in class. One of the key elements is to select a backwards light cone with a large enough propagation speed. Note that it is not possible to do this by looking at the energy on all of $\mathbb{R}^n$ (why?). You may find it useful to prove the following inequality for all vectors $v, w \in \mathbb{R}^n$ (which uses just that $A(x) \geq 0$),

$$2\langle A(x)v, w\rangle \leq \langle A(x)v, v\rangle + \langle A(x)w, w\rangle.$$

Problem 2. Shearer and Levy: Chapter 4, Problem 6.

Problem 3. Let u be the D'Alembert solution of the 1-d wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in} \quad \mathbb{R} \times (0,\infty) \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x) & \text{in} \quad \mathbb{R}. \end{cases}$$

Under what condition on the initial data ϕ, ψ will u be a travelling wave moving to the right (i.e. $u(x,t) = F(x - ct)$ for some F)? Under what condition will it be a travelling wave moving to the left?

Problem 4. [Evans 2nd Ed., Ch. 2 problem 23] Let S denote the square lying in $\mathbb{R} \times (0,\infty)$ with corners at the points $(0,1)$, $(1,2)$, $(0,3)$ and $(-1,2)$. Define,

$$f(x,t) := \begin{cases} -1 & \text{for} \quad (x,t) \in S \cap \{t > x+2\} \\ 1 & \text{for} \quad (x,t) \in S \cap \{t < x+2\} \\ 0 & \text{else} \end{cases}$$

Assume that u solves,

$$\begin{cases} u_{tt} - u_{xx} = f(x,t) & \text{in} \quad \mathbb{R} \times (0,\infty) \\ u = 0,\ u_t = 0 & \text{on} \quad \mathbb{R} \times \{t = 0\}. \end{cases}$$

Describe the shape of u for times $t \geq 3$.

Hint: You should use the Duhamel's formula for the wave equation (which I will derive in class on Tuesday) which in this case says,

$$u(x,t) = \int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,s)\ dyds = \int_{\Delta(x,t)} f(y,s)\ dyds.$$

Here $\Delta(x,t) = \{(y,s) \in \mathbb{R} \times (0,\infty) : x - (t-s) \leq y \leq x + (t-s)\}$ is the backwards light cone from (x,t).