MATH 275: Homework 3

Due Thursday, April 21

Problem 1. [Poincare inequality and the first Dirichlet eigenvalue] Consider minimizing the Dirichlet Energy,

$$I[u] = \int_0^1 u'(x)^2 \, dx \quad \text{over the class} \quad \mathcal{A} = \{u \in C^2([0,1]) : \; u(0) = u(1) = 0 \; \text{ and } \; \int_0^1 u^2 \, dx = 1\}.$$

For this problem you may assume the existence of a minimizer $u \in \mathcal{A}$ so that $u = \text{argmin}_{w \in \mathcal{A}} I[w]$, in problem 2 part (d) you will show that $\inf_{w \in \mathcal{A}} I[w] > 0$ which makes the existence of a minimizer more plausible.

(a) Show that if $u \in \mathcal{A}$ satisfies,

$$I[u] = \min_{w \in \mathcal{A}} I[w] \quad \text{then } u \text{ solves} \quad \begin{cases} -u'' = \lambda u & \text{in} \quad (0,1) \\ u(0) = u(1) = 0 & \text{with} \quad \lambda = I[u]. \end{cases} \tag{1}$$

Hint: As in class you should try to perturb the minimizer u to $u + \varepsilon\varphi$, this isn't in the admissible class $\mathcal{A}$ but you can fix that by looking at $(u+\varepsilon\varphi)/(\int_0^1 |u+\varepsilon\varphi|^2 dx)^{1/2}$ instead.

(b) By solving the ODE boundary value problem in (1) find the value of

$$\rho = \min_{w \in \mathcal{A}} I[w] > 0.$$

Hint: Start by solving the initial value problem $-u'' = \lambda u$ with $u(0) = 0$, you will find that in order to have a non-zero solution of the boundary value problem, $\lambda > 0$ will be forced to lie in a discrete set of values.

(c) Show that for every $f \in C^2([0,1])$ with $f(0) = f(1) = 0$,

$$\int_0^1 f(x)^2 \, dx \le \frac{1}{\rho} \int_0^1 f'(x)^2 \, dx.$$

Problem 2. [Long time behavior - interval w/ Dirichlet BC] Let $g : [0,1] \to \mathbb{R}$ be continuous with $g(0) = g(1) = 0$ and let $u \in C^\infty((0,1) \times (0,\infty)) \cap C([0,1] \times [0,\infty))$ be a solution of the initial/boundary value problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{in} \quad (0,1) \times (0,\infty) \\ u(x,0) = g(x) & \text{in} \quad [0,1] \\ u(0,t) = u(1,t) = 0 \end{cases} \tag{2}$$

(a) Define the energy functional,

$$E(t) = \int_0^1 u(x,t)^2 \, dx,$$

making use of the result of Problem 1 show that,

$$E(t) \le E(0) e^{-2\rho t}.$$

Hint: Start by computing $E'(t)$.

(b) Show that the rate of convergence established in the previous problem is optimal, i.e. give an example of a g so that the solution $u(x,t)$ of (2) satisfies,

$$E(t) \ge E(0) e^{-2\rho t}.$$

Hint: Look for a solution of the form $u(x,t) = v(x)\tau(t)$, try using the solution you found in Problem 1 part (a) for $v(x)$.

(c) Let f, g be square-integrable functions on an open set $U \subset \mathbb{R}^n$ show the Cauchy-Schwarz inequality,

$$|\int_U f(x) g(x) \, dx| \le (\int_U |f(x)|^2 \, dx)^{1/2} (\int_U |g(x)|^2 \, dx)^{1/2}.$$

Hint: Start from $\int_U (\lambda f - g)^2 \, dx \ge 0$ and optimize the resulting inequality in λ.

(d) Show that so that for any $f \in C^1([0,1])$ and any $x \in [0,1]$,

$$|f(x) - \int_0^1 f(y)\ dy| \le (\int_0^1 |f'(y)|^2\ dy)^{1/2}.$$

If additionally $f(0) = f(1) = 0$ show that for all $x \in [0,1]$,

$$|f(x)| \le \frac{1}{\sqrt{2}}(\int_0^1 |f'(y)|^2\ dy)^{1/2}.$$

(e) Use the first result of part (d) to show that the Dirichlet energy,

$$D(t) = \int_0^1 u_x(x,t)^2\ dx,$$

satisfies

$$D(t) \le D(0)e^{-2t}.$$

Use the second result of part (d) to show that,

$$\sup_{x\in[0,1]} |u(x,t)| \le \frac{1}{\sqrt{2}} D(t)^{1/2} \le \frac{1}{\sqrt{2}} D(0)^{1/2} e^{-t}.$$

As a result we obtain that the solution of the Dirichlet problem (2) converges to 0 with exponential rate.

Problem 3. [Long time behavior - real line] Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function with

$$\lim_{x\to-\infty} g(x) = a \quad \text{and} \quad \lim_{x\to+\infty} g(x) = b.$$

Let $u(x,t)$ be the solution of the heat equation on $\mathbb{R} \times (0,\infty)$ given by,

$$u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t)g(y)\ dy \quad \text{with} \quad \Phi(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}.$$

Show that for every $R > 0$

$$\sup_{|x|\le R} |u(x,t) - \frac{a+b}{2}| \to 0 \quad \text{as} \quad t \to \infty.$$