MATH 275: Homework 2

Due Thursday, April 14

Note: A domain in $\mathbb{R}^n$ is an *open* and *connected* subset of $\mathbb{R}^n$.

Problem 1. Let U be a bounded domain of $\mathbb{R}^n$ and $u \in C^2(U) \cap C(\overline{U})$ which is harmonic in U. Suppose that $u(x_0) = \min_{\overline{U}} u = 0$ at some $x_0 \in \partial U$. Suppose that U has an interior tangent ball at x_0, that is there exists x_1 so that $B(x_1, r) \subset U$ and $\partial B(x_1, r) \cap \partial U = \{x_0\}$. Prove that if u is not constant then,

$$\frac{\partial u}{\partial \nu}(x_0) < 0,$$

where ν is the outward unit normal to $B(x_1, r)$ at x_0.

Note: I have not assumed enough to guarantee that the normal derivative $\frac{\partial u}{\partial \nu}(x_0)$ actually exists so take the problem statement to mean,

$$\limsup_{h \to 0^+} \frac{u(x_0) - u(x_0 - h\nu)}{h} < 0.$$

Hints: First use strong maximum principle to conclude that $u > 0$ in U or u is constant. If $u > 0$ in U try to show that $u(x) \geq c(|x - x_1|^{2-n} - r^{2-n})$ in $B_r(x_1) \setminus B_{r/2}(x_1)$ for some small $c > 0$.

Problem 2. Let U be a bounded domain of $\mathbb{R}^n$ with C^2 boundary, in particular it has an interior tangent ball at every boundary point. Using the result of the previous problem show that any two solutions of the Neumann problem

$$\begin{cases} -\Delta u = f & \text{in} \quad U \\ \frac{\partial u}{\partial \nu} = h & \text{on} \quad \partial U \end{cases} \tag{N}$$

differ by a constant.

Problem 3. Computing Green's functions for ODE boundary value problems.

(a) Let $k > 0$, find the Green's function associated with the boundary value problem

$$-u'' + k^2 u = f(x) \quad \text{for} \quad x \in (0, \infty) \quad \text{with} \quad u(0) = 0, \ \lim_{x \to \infty} u(x) = 0.$$

(b) Let $p : [0, 1] \to \mathbb{R}$ be strictly positive and continuous. Find the Green's function associated with the boundary value problem,

$$-(p(x)u')' = f(x) \quad \text{for} \quad x \in (0, 1) \quad \text{with} \quad u(0) = 0, \ u(1) = 0.$$

Hint: Remember my description from class of how to solve for $G(x, y)$. Fix x and look for two solutions of the ODE u_L and u_R, you will eventually set $G(x, y) = u_L(y)$ when y is to the left of x and $G(x, y) = u_R(y)$ when y is to the right of x. Since you have two solutions of a second order ODE there will be *four* constants of integration to specify. Two will be used to fix the boundary data. The third will be used to make $G(x, y)$ continuous at x and the final constant will be chosen so that $\frac{\partial G}{\partial y}$ has a jump of the correct height at x.

Problem 4. Prove that a harmonic function $u : \mathbb{R}^n \to \mathbb{R}$ with *subquadratic growth*, i.e.

$$\lim_{R \to \infty} \left[\sup_{x \in \partial B(0,R)} \frac{|u(x)|}{R^2} \right] = 0,$$

is linear, i.e. there is $p \in \mathbb{R}^n$ and $c \in \mathbb{R}$ so that $u(x) = p \cdot x + c$.

Problem 5. Let U be a bounded domain of $\mathbb{R}^n$ and $b : \overline{U} \to \mathbb{R}^n$ be continuous. Prove that there is at most one solution $u \in C^2(U) \cap C(\overline{U})$ of the Dirichlet problem,

$$\begin{cases} -\Delta u + b(x) \cdot \nabla u = f(x) & \text{in} \quad U \\ u = g(x) & \text{on} \quad \partial U. \end{cases}$$

Hint: Prove a weak maximum principle for solutions of $-\Delta u + b(x) \cdot \nabla u = 0$. You should try to follow the second method I used in lecture, first show the maximum principle for *strict subsolutions* satisfying $-\Delta u + b(x) \cdot \nabla u < 0$. Next, for non-strict subsolutions (e.g. solutions), you will need to make a perturbation like we did in class, using $v(x) = \delta|x|^2$ will not work anymore though so you will need to find a better function to perturb by. I recommend to look for a perturbing function $v(x_1)$ (as opposed to looking for something radial).

Problem 6. Let U be a bounded domain with smooth boundary. Let $u \in C^2(U) \cap C(\overline{U})$ be harmonic in U. Show that for any other function $v \in C^2(U) \cap C(\overline{U})$ with $v = u$ on ∂U,

$$\int_U |Du|^2 \, dx \leq \int_U |Dv|^2 \, dx.$$