**Some Background on Theory of ODE Initial Value Problems**
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In this note I will explain some basic theory of ODE including existence of solutions, uniqueness and continuous dependence on the initial data. We consider the solution $X : [0, \infty) \to \mathbb{R}^d$ of the following initial value problem (IVP) with initial data $x_0 \in \mathbb{R}^d$ and initial time $t_0 \in \mathbb{R}$,

$$\begin{cases} \dot{X}(t) = F(X(t), t) \ \text{ for } \ t > t_0 \\ X(t_0) = x_0 \end{cases} \tag{0.1}$$

Here $F : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^d$ would typically be at the least a continuous function in both variables. We will be interested in examining the well-posedness of the ODE system, we would like to answer the following questions which will again be extremely important for PDE as well.

- $(i)$ Local/global in time existence of solutions – is there a solution of (0.1) on (local) a small interval of times containing $t_0$, (global) all $t \in \mathbb{R}$?
- $(ii)$ Uniqueness – is there only one solution of (0.1)?
- $(iii)$ Continuous dependence / stability – does the solution $X(t)$ depend continuously on the initial data $x_0$?

As we will see below continuity of $F$ is sufficient to obtain the existence of a *local in time* solution of the ODE IVP (0.1). For uniqueness and global existence we will show that *Lipschitz continuity* suffices.

When uniqueness does hold we can think of the solution of the ODE system as a *flow* on $\mathbb{R}^d$ induced by the *velocity field* $F$. It is often conceptually useful (especially when we study the *method of characteristics* which can be used to solve first order PDE) to write $X(t; x_0)$ to make clear the dependence of the (unique) solution on the initial data. For each $t \geq 0$, $X(t; \cdot)$ is a mapping of $\mathbb{R}^d$ to itself, which to each $x_0$ assigns the location at time $t$ of the trajectory started at $x_0$ at time $t_0$.

It can be very useful, especially for the purposes of existence and uniqueness proofs, to introduce an equivalent *integral equation* associated with the ODE initial value problem (0.1). It is not too difficult to check that any, say $C^1$, solution of (0.1) also satisfies,

$$X(t) = x_0 + \int_{t_0}^{t} F(X(s), s) \, ds \ \text{ for all } \ t \geq t_0. \tag{0.2}$$

It is also easy to check that any continuous $X$ which satisfies (0.2) is also $C^1$ and solves the original differential equation (0.1). It is worth noting that while (0.1) seems to require differentiability of $X$ to make sense of, the equivalent integral form of the equation requires much less regularity – for example continuity is enough – to make sense.

0.1. **Short time existence.** This is usually called the Cauchy-Peano existence theorem.

**Theorem 0.1.** *Let $F$ be continuous on an open set $\mathcal{O} \subset \mathbb{R}^d \times \mathbb{R}$. For each $(x_0, t_0) \in \mathcal{O}$ there is $\varepsilon > 0$ so that there exists a $C^1$ solution $X(t; x_0)$ of (0.1) on $(t_0 - \varepsilon, t_0 + \varepsilon)$.*

*Proof.* The proof uses the integral form of the ODE. By continuity of $F$ there is $\delta > 0$ so that $\Gamma_\delta = \overline{B}_\delta(x_0) \times [t_0 - \delta, t_0 + \delta] \subset \mathcal{O}$ and,

$$|F(x, t) - F(x_0, t_0)| \leq 1 \ \text{ for } \ (x, t) \in \Gamma_\delta.$$

We define a successive approximation scheme,

$$X_0(t) = x_0 \ \text{ and } \ X_k(t) = x_0 + \int_{t_0}^{t} F(X_{k-1}(s), s) \, ds \ \text{ for } \ k \geq 1. \tag{0.3}$$

To obtain a convergence subsequence we need to show boundedness and equicontinuity. Since $\Gamma_\delta$ is a closed set and $F$ is a continuous function on that set there is $M > 0$ so that $|F(x, t)| \leq M$ on $\Gamma_\delta$. Let

$\varepsilon = \min\{\delta, \delta/M\}$ we claim that on time interval $[t_0 - \varepsilon, t_0 + \varepsilon]$ all of the $(X_k(t), t)$ lie in $\Gamma_\delta$, this is true for $X_0$ and then proceeding inductively we have,

$$|X_k(t) - x_0| \leq \left| \int_{t_0}^{t} |F(X_{k-1}(s), s)| \, ds \right| \leq M|t - t_0| \leq M\varepsilon \leq \delta,$$

where we have used the inductive hypothesis $(X_{k-1}(s), s) \in \Gamma_\delta$ to bound the integrand.

Now we need to check equicontinuity, this will also follow from the integral nature equation for $X_k$. Uniform boundedness of the derivative leads equicontinuity of the function. This is an idea which will come up a lot in PDE, for the solutions of differential equations unlike for generic functions uniform bounds on the function itself often lead to uniform bounds on derivatives through the equation. Using that $(X_k(t), t) \in \Gamma_\delta$ for all $k$ and $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ we obtain that,

$$|\dot{X}_k(t)| = |F(X_{k-1}(t), t)| \leq M \quad \text{for all } k, \; t \in [t_0 - \varepsilon, t_0 + \varepsilon].$$

Thus $X_k$ are a bounded and equicontinuous so by the Arzela-Ascoli Theorem there is a subsequence $X_{n_k}$ which converges uniformly to a function $X(t)$ on $[t_0 - \varepsilon, t_0 + \varepsilon]$. Now we just need to check that $X(t)$ is in fact a solution of the equation, this follows by taking the limit on both sides of (0.3).

□

0.2. **Important counter-examples.** Now that we have proven the existence of a solution for a short time it is necessary to start thinking about what an go wrong. Our existence proof gave no indication of the uniqueness of the solution, continuous dependence of data or about how long is the time of existence. In fact it turns out that the assumptions we have put so far are not sufficient to guarantee uniqueness (and hence continuous dependence fails as a consequence) or existence of a global in time solution. Let us start by considering the ODE initial value problem,

$$\dot{X} = X^{1/3} \quad \text{with} \quad X(0) = 0. \tag{0.4}$$

We can start off by noticing that $X(t) \equiv 0$ is a solution. The equation is separable so we can integrate it to obtain two more solutions,

$$X(t) = \pm(\tfrac{2}{3}t)^{3/2}.$$

But now, noticing that for the solutions above $\dot{X}(0) = 0$ and using the time translation invariance of the equation, we can patch together the solutions above to obtain an infinite family of solutions. For any $t_0 \geq 0$,

$$X_{t_0,\pm}(t) = \begin{cases} 0 & t \leq t_0 \\ \pm(\frac{2}{3})^{3/2}(t - t_0)^{3/2} & t > t_0. \end{cases}$$

The key thing to note again is that this piecewise defined function is in fact $C^1$ and satisfies the ODE pointwise in addition to having the correct initial data.

Another important example, which is a common enemy in showing global existence in PDE problems, is the ODE

$$\dot{X} = X^2 \quad \text{with} \quad X(0) = 1.$$

This is again a separable equation and we can integrate it to obtain,

$$X(t) = \frac{1}{1 - t} \quad \text{for} \quad t < 1.$$

There is a (unique) local in time solution for every initial data, but besides $X(t) \equiv 0$ all the solutions with non-negative initial data go to $+\infty$ in a finite time. This is an important phenomenon which will occur in PDE as well called *finite time blow-up.*

0.3. **Grönwall inequality.** We now introduce a very useful and important inequality which translates an integral inequality (or a differential inequality) for a quantity $y(t)$ into a bound on $y(t)$ itself.

**Theorem 0.2.** *Suppose that there are $\alpha \in \mathbb{R}$ and $\beta > 0$ so that $y : [0, \infty) \to \mathbb{R}$ a continuous function satisfies for every $t \in \mathbb{R}$*

$$y(t) \leq \alpha + \beta \int_0^t y(s)\ ds$$

*then*

$$y(t) \leq \alpha e^{\beta t} \quad \textit{for every } t \in \mathbb{R}.$$

*Proof.* The idea is to translate the integral inequality into an easier to handle differential inequality, $y(t)$ isn't necessarily differentiable so the trick is to instead consider,

$$Y(t) = \alpha + \beta \int_0^t y(s)\ ds.$$

This quantity $Y$ is $C^1$ since $y$ is continuous and so we can compute,

$$\dot{Y}(t) = \beta y(t) \leq \beta(\alpha + \beta \int_0^t y(s)\ ds) = \beta Y(t).$$

This is the same as

$$\frac{d}{dt}(e^{-\beta t} Y) \leq 0.$$

Integrating the above inequality from 0 to $t$,

$$e^{-\beta t} Y(t) - Y(0) \leq 0.$$

Finally we use that $Y(0) = \alpha$ to obtain,

$$Y(t) \leq \alpha e^{\beta t}.$$

□

0.4. **Uniqueness and continuous dependence.** It is quite natural to prove uniqueness and continuous dependence at the same time. We will omit the $t$-dependence in the right hand side $F$ for simplicity.

**Theorem 0.3.** *Suppose that $F : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous with constant $K > 0$, i.e. for every $x, y \in \mathbb{R}^d$ it holds that $|F(x) - F(y)| \leq K|x - y|$. If $X$ and $Y$ are two solutions of $\dot{X} = F(X)$ and $\dot{Y} = F(Y)$ then*

$$|X(t) - Y(t)| \leq e^{Kt} |X(0) - Y(0)| \quad \textit{for all} \quad t > 0.$$

*In particular if $X(0) = Y(0)$ then $X(t) = Y(t)$ for all $t > 0$.*

*Proof.* Using the integral form of the equation,

$$\begin{aligned} |X(t) - Y(t)| &\leq |X(0) - Y(0)| + \int_0^t |F(X(s)) - F(Y(s))|\ ds \\ &\leq |X(0) - Y(0)| + K \int_0^t |X(s) - Y(s)|\ ds. \end{aligned}$$

Then we can apply Grönwall's inequality to get the desired result. □

0.5. **Problems.** I am not asking you to turn these problems in but you may want to try them out to practice your understanding of the material.

1. Consider the ODE IVP,
$$\dot{X} = F(X) \quad \text{with} \quad X(0) = x_0. \tag{0.5}$$
Suppose that $F$ is locally Lipschitz continuous on an open set $\mathcal{O} \subseteq \mathbb{R}^d$ and $x_0 \in \mathcal{O}$. Show that there exists a maximal time $T(x_0)$ possibly equal to $+\infty$ so that there is a solution of (0.5) existing on the interval $[0, T(x_0))$ and, if $T(x_0) < +\infty$, then,
$$\lim_{t \nearrow T} X(t) \notin \mathcal{O}.$$