The Wave Equation

- time reversible
- energy conserving
- finite speed of propagation

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mho \times (0,\infty) \\ \text{boundary conditions} & \text{(Dirichlet, Neumann)} \end{cases}$$

$u(x,t)$ is displacement of a string (1-d) or elastic membrane (drumhead) (2-d)

or pressure (sound waves in a gas) 3-d

conserves energy

$$E(t) = \frac{1}{2}\int_{\mho} \underbrace{u_t^2}_{\text{kinetic}} + \underbrace{|Du|^2}_{\text{~~potential~~ elastic}}\, dx$$

& if u solves wave eqn w/ 0 dirichlet data on $\partial\mho \times (0,T]$

$$\frac{d}{dt}E(t) = \int_{\mho} u_t u_{tt} + Du \cdot Du_t \, dx$$

$$= \int_{\mho} u_t(u_{tt} - \Delta u)\, dx + \cancel{\int_{\partial\mho} u_t \frac{\partial u}{\partial \nu}\, dS_x}$$

($u = 0$ on $\partial\mho$ $\Rightarrow u_t = 0$ on $\partial\mho$)

$= 0$ since $u_{tt} - \Delta u = 0$ in $\mho$

Wave Eqn in 1-d : D'Alembert's Solution

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in } \mathbb{R}\times(0,\infty) \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) & \text{in } \mathbb{R} \end{cases}$$

factor $0 = (\partial_{tt} - c^2 \partial_{xx}) u = (\partial_t - c\partial_x)(\partial_t + c\partial_x) u$

and change variables to moving reference frames

$$\xi = x + ct \qquad \eta = x - ct$$

$$\partial_\xi = \frac{1}{2c}(\partial_t + c\partial_x) \qquad \partial_\eta = \frac{1}{2c}(\partial_t - c\partial_x)$$

so $\partial^2_{\xi\eta} u = 0$ implies that

$$u(x,t) = F(\eta) + G(\xi) = F(x-ct) + G(x+ct)$$

is general form of solution

u is a superposition of travelling waves

profile of F moves to the right w/ speed c

G moves to the left w/ speed c

now how to solve the Cauchy Problem

$$(1) \quad \begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x) & \text{in } \mathbb{R} \\ u_t(x,0) = g(x) & \end{cases}$$

<u>Thm</u> If f is C^2 and g is C^1 the unique solution of (1) is

$$u(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$$

proof: The general solution of (1) is

$$u(x,t) = F(x-ct) + G(x+ct)$$

$$u(x,0) = F(x) + G(x) = f(x)$$
$$u_t(x,0) = -cF'(x) + cG'(x) = g(x)$$

so $$-F(x) + G(x) = \frac{1}{c}\int_0^x g(y)\,dy + A$$

solving the 2x2 linear system

$$-2F(x) + f(x) = \frac{1}{c}\int_0^x g(y)\,dy + A$$

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(y)\,dy - \frac{A}{2}$$

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(y)\,dy + \frac{A}{2}$$

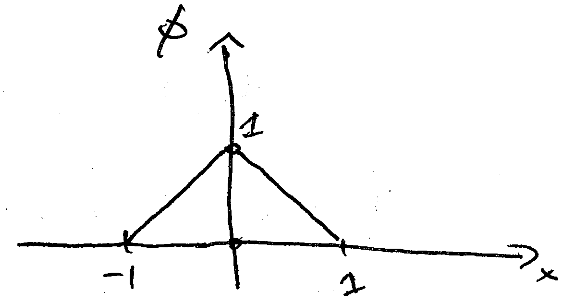
So
$$u(x,t) = F(x-ct) + G(x+ct)$$
$$= \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_0^{x+ct} g(y)\,dy - \frac{1}{2c}\int_0^{x-ct} g(y)\,dy$$
$$= \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$$

$\square$

This is called D'Alembert's solution of the wave equation.

<u>Example 1</u> Let's take $c=1$ and $\psi = 0$

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}\times(0,\infty) \\ u(x,0) = \phi(x) \quad u_t(x,0) = 0 & \text{in } \mathbb{R} \end{cases}$$

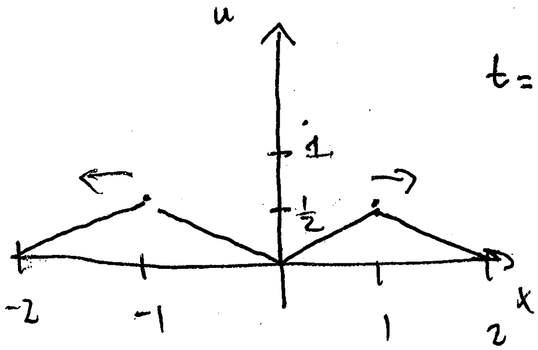


$\cancel{\phi(x,t) = (x-1)}$

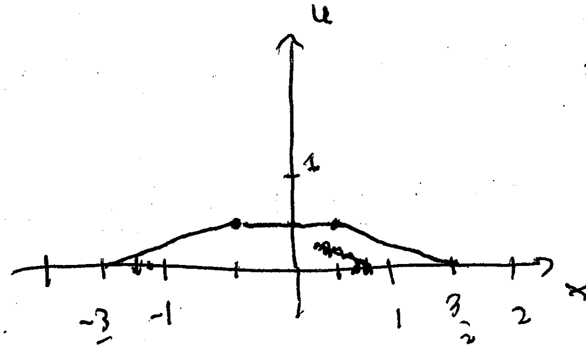
$$u(x,t) = \frac{1}{2}\left(\phi(x+t) + \phi(x-t)\right)$$

$\frac{1}{2}\phi(x+t) \rightarrow \frac{1}{2}\phi$ translated to the left by t

$\frac{1}{2}\phi(x-t) \rightarrow \frac{1}{2}\phi$ " " " right " t



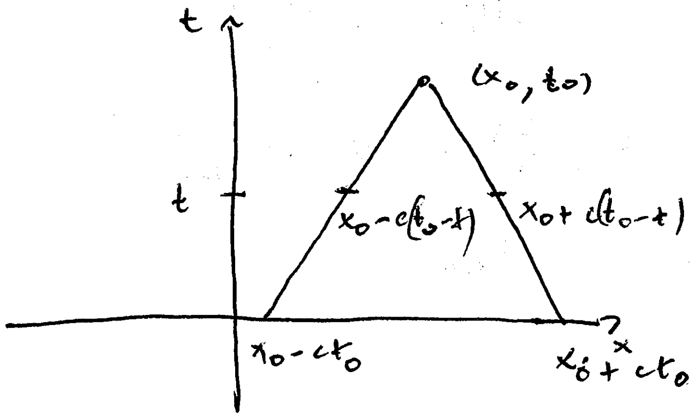
$t = 1$



$t = \frac{1}{2}$

Domain of dependence & region of influence

(Light cones)



~~$x_0 - c(t - t_0)$~~

~~$x_0 + c(t - t_0)$~~

these lines are called backwards characteristics

$(x_0 - ct_0, x_0 + ct_0)$ is known as the interval of dependence

the entire domain of dependence ~~triangle~~ is called the cone of (x_0, t_0)

or the backwards light cone from (x_0, t_0)

~~In higher dimensions~~

it is clear from D'Alembert's formula.

that

~~What~~

Thm let $\phi_1, \psi_1, \phi_2, \psi_2$ and corresponding u_1, u_2

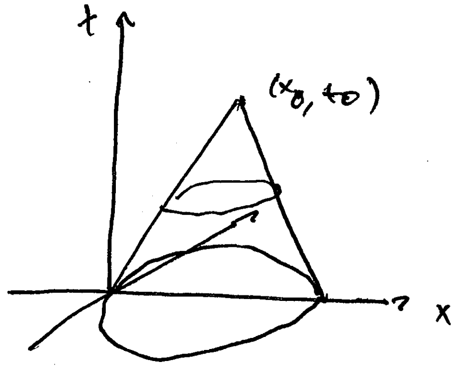
solving
$$\begin{cases} \partial_{tt}^2 u_j - \partial_{xx}^2 u_j = 0 & \text{in } \mathbb{R}\times[0,\infty) \\ u_j(x,0) = \phi_j(x) \quad \partial_t u_j(x,0) = \psi_j(x) & \text{in } \mathbb{R} \end{cases}$$

If $\cancel{\phi_1 = \phi_2}$ $(\phi_1, \psi_1) = (\phi_2, \psi_2)$

in $(x_0 - ct, x_0 + ct)$ then

$u_1(x_0, t_0) = u_2(x_0, t_0)$.

<u>Higher Dimensions</u>



<u>Cone of dependence</u>

$$\Delta(x_0, t_0) = \{ (x,t) \in \mathbb{R}^n \times [0,\infty) : |x - x_0| \le c(t_0 - t) \}$$

Cone <u>of influence</u>

$$\{ (x,t) \in \mathbb{R}^n \times [0,\infty) : |x - x_0| \le c(t - t_0) \}$$

u^j solve

$$\begin{cases} u^j_{tt} - \Delta u^j = f^j & \text{in } \mathbb{R}\times[0,\infty) \\ u^j(x,0) = \phi^j(x) \quad u^j_t(x,0) = \psi^j & \text{in } \mathbb{R} \end{cases} \qquad j = 1,2$$

Thm If $(\phi_1, \psi_1, f_1) = (\phi_2, \psi_2, f_2)$ in $\Delta(x_0, t_0)$

then $u^1(x_0, t_0) = u^2(x_0, t_0)$

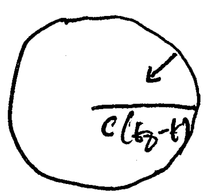
proof Call $u = u^1 - u^2$

satisfies
$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \Delta(x_0, t_0) \\ u(x,0) = 0, \quad u_t(x,0) = 0 & \text{in } |x - x_0| \le ct_0 \end{cases}$$

Consider the energy in the light cone

$$E(t) = \frac{1}{2}\int_{|x-x_0| \le ct} u_t^2 + c^2|Du|^2 \, dx, \quad E(0) = \frac{1}{2} 0$$

$$E'(t) = \int_{|x-x_0| < c(t_0-t)} u_t u_{tt} + c^2 \cancel{|Du|^2} Du \cdot Du_t \, dx$$
$$+ \int_{\partial |x-x_0| = c(t_0-t)} \frac{1}{2}(-c) \cdot (u_t^2 + c^2|Du|^2) \, dS_x$$



boundary moving in w/ speed c.

$$= \int_{|x-x_0| < c(t_0-t)} u_t \Delta u - u_t \Delta u \, dx$$
$$+ \int_{|x-x_0| = c(t_0-t)} c^2 u_t \frac{\partial u}{\partial \nu} - \frac{1}{2} c (u_t^2 + c^2|Du|^2) \, dS_x$$

$$= 0 + \int_{|x-x_0| = c(t_0-t)} c \left(u_t \left(c \frac{\partial u}{\partial \nu} \right) - \frac{1}{2}(u_t^2 + c^2|Du|^2) \right) dS_x$$

now using $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$

obtain

$$\le c\int_{|x-x_0|=c(t_0-t)} \frac{1}{2} u_t^2 + \frac{1}{2} c^2\left(\frac{\partial u}{\partial \nu}\right)^2 - \frac{1}{2} u_t^2 - \frac{1}{2} |Du|^2 \, dS_x$$

so using that $\left|\frac{\partial u}{\partial \nu}\right| = |Du\cdot\nu| \le |Du|$.

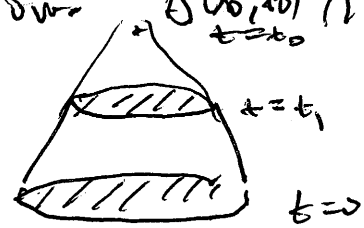
so $\quad E'(t) \le 0 \quad \Rightarrow 0 \le E(t) \le E(0) = 0$

$E(t) = 0 \quad$ for $\quad 0 \le t \le t_0$.

<u>Alternative method:</u>

$$u_{tt} - c^2 \Delta u = 0 \quad \text{in} \quad \Delta(x_0, t_0)$$

multiply by u_t and integrate over $\Delta(x_0,t_0) \cap \{t \le t_1\}$



$$0 = \int_{\Delta(x_0,t_0)\cap\{t\le t_1\}} u_t u_{tt} - c^2 u_t \Delta u \, dx\, dt$$

$$= \int_{\Delta(x_0,t_0)\cap\{t\le t_1\}} \partial_t\left(\frac{u_t^2}{2} + \frac{c^2}{2}|Du|^2\right) - \nabla\cdot(c^2 u_t \nabla u)\, dx\, dt$$

$$= \int_{\Delta(x_0,t_0)\cap\{t\le t_1\}} (\partial_t, \nabla)\cdot\left(\frac{u_t^2}{2} + \frac{c^2}{2}|Du|^2, -c^2 u_t Du\right) dx\, dt$$

these have names

$$e(x,t) = \tfrac{1}{2}\left(u_t^2 + c^2 |\nabla u|^2\right) \qquad \underline{\text{energy density}}$$

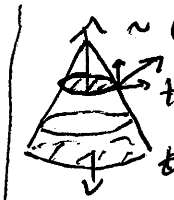
$$p(x,t) = -c^2 u_t \nabla u \qquad \underline{\text{momentum density}}$$

$$= \int_{\Delta(x_0,t_0)\cap\{t\le t_1\}} (\partial_t, \nabla)\cdot(e,p)\,dx\,dt$$

(divergence thm)

$$= \int_{\partial\Delta(x_0,t_0)} (n_t, n_x)\cdot(e,p)\,dx\,dt$$

$\partial(\Delta(x_0,t_0)\cap\{t\le t_1\})$ $\cancel{\partial\Delta(x_0,t_0)}$ =

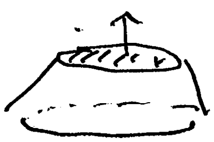


$\sim \left(c, \frac{x-x_0}{|x-x_0|}\right)$ $t_1 = t$ $0 = t$

cone $t>0$ $|x-x_0| = c(t_0-t)$

$(n_t,n_x) = \frac{1}{\sqrt{1+c^2}}\left(c, \frac{x-x_0}{|x-x_0|}\right)$

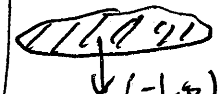
and



$t = t_1$ &

$|x-x_0| \le c(t_0-t_1)$

$(n_t,n_x) = (1,0)$



$(-1,0)$

initial disk $t=0$ $|x-x_0| \le ct_0$

$(n_t,n_x) = (-1,0)$

outward unit normal

$[e(x,0) = 0$ in $|x-x_0|\le ct_0]$

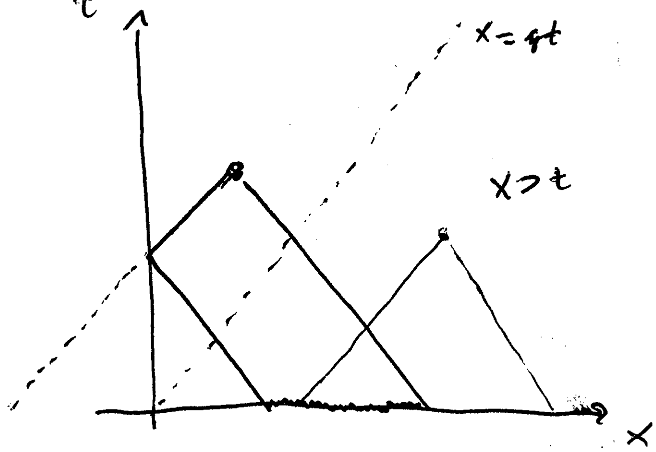
$$= \cancel{\int_{|x-x_0|\le ct_0} (-1,0)\cdot(e,p)\,dx} + \int_{\substack{|x-x_0|\le c(t_0-t_1)\\ t=t_1}} (1,0)\cdot(e,p)\,dx$$

$$+ \frac{1}{\sqrt{1+c^2}} \underbrace{\int_{\substack{|x-x_0| = c(t_0-t)\\ 0<t<t_0}} c\,e(x,t) - \frac{x-x_0}{|x-x_0|}\cdot p \, dS_{x,t}}_{\ge 0 \text{ by same argument as before}}$$

So $\int_{|x-x_0|\le c(t_0-t_1)} e(x,t_1)\,dx \le 0 \Rightarrow \boxed{u\equiv 0 \text{ in } \Delta(x_0,t_0)}$

Wave Eqn in 1-d w/ a Boundary

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in } x>0,\ t>0 \\ u(0,t) = 0 & \text{for } t>0 \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x) & \text{for } x>0 \end{cases}$$



for $x > ct$

solution is same as if boundary was not present.

$$u(x,t) = \frac{1}{2}\left(\phi(x+t) + \phi(x-t)\right) + \frac{1}{2c}\int_{x-t}^{x+t} \psi(y)\,dy$$

for $x > t$

for $x < t$

Think of extending ϕ, ψ to be odd fcns on $\mathbb{R}$ $\phi(-x) = -\phi(x)$, $\psi(-x) = -\psi(x)$.

w/ solution of wave eqn $w(x,t)$

then ~~$u(-x,t)$~~ $v(x,t) = -w(-x,t)$

solves wave equation w/ same initial data as u. Uniqueness

$\Rightarrow$ $w(x,t) = \underbrace{\cancel{w(x,t)}}_{v(x,t)} = -w(-x,t)$

So initial data odd

$\Rightarrow$ $w(\cdot,t)$ is ~~and~~ odd fcn

$\Rightarrow$ $w(0,t) = 0$ for all $t > 0$.

So again by uniqueness

$u(x,t) = w(x,t)$ for $x > 0$, $t > 0$.

but from formula for w

$x > t$

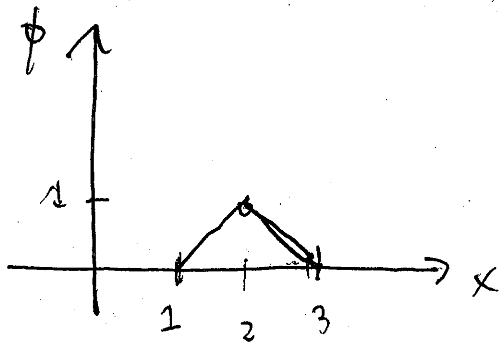
$$u(x,t) = \frac{1}{2}(\phi(x+t) + \phi(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} \psi(y)\,dy$$

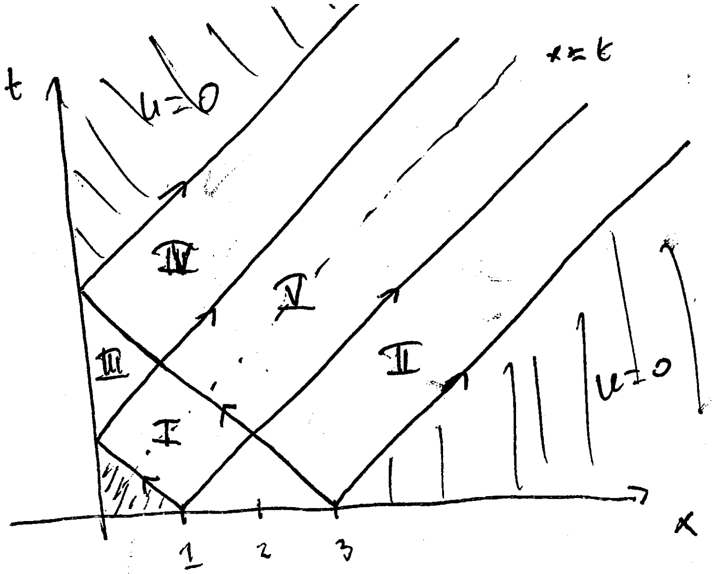
$$= \frac{1}{2}(\phi(x+t) - \phi(t-x)) + \cancel{\frac{1}{2}\int_{t-x}^{x+t} \psi(y)\,dy}$$

$$+ \underbrace{\frac{1}{2}\left[\int_0^{x+t} \psi(y)\,dy - \int_0^{t-x} \psi(y)\,dy\right]}_{= \frac{1}{2}\int_{t-x}^{t+x} \psi(y)\,dy}$$

Example

$\psi = 0$

$\phi =$ 



$(a,b) = (1,3)$

region I: ~~x+t~~ $x-t < a$ and $t-x < a$

so

$$u(x,t) = \frac{1}{2}\phi(x+ct) + \frac{1}{2}\int_a^{x+t} \psi(y)\,dy$$

left moving wave

region II: $x-t \in (a,b)$, $x+t > b$ so

$$u(x,t) = \frac{1}{2}\phi(x-t) + \frac{1}{2}\int_{x-t}^{b} \psi(y)\,dy$$

the right moving wave

region III: left moving wave is partially reflected

region IV: $t-x \in (a,b)$ and $x+t > b$ so

$$u(x,t) = -\frac{1}{2}\phi(t-x) + \frac{1}{2}\int_{t-x}^{b} \psi(y)\,dy$$

wave profile from the initial data has reversed sign and is moving to the right.

Region V

$t-x < a < b < x+t$

so $u(x,t) = \frac{1}{2}\int_a^b \psi(y)\,dy$

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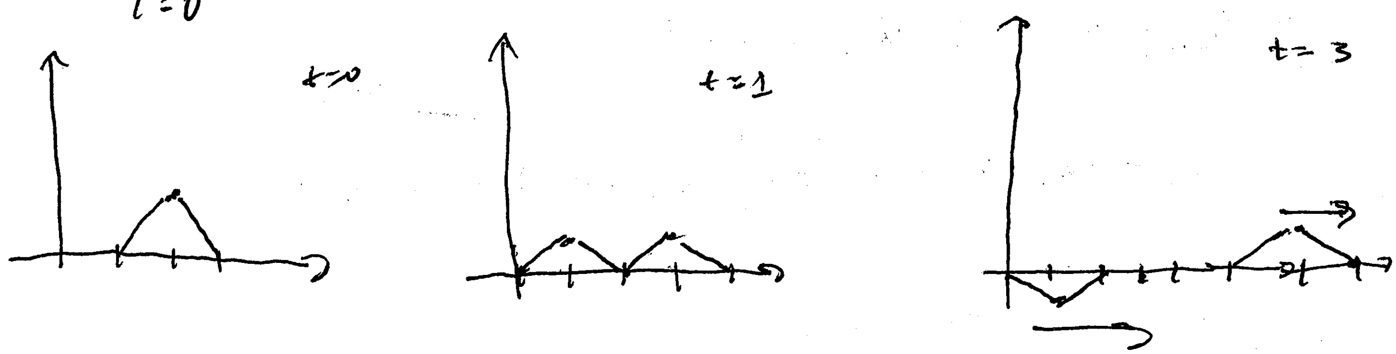
if $\psi = 0$



Duhamel's principle (in 1-d)

$$(1)\quad \begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & x \in \mathbb{R},\ t>0 \\ u(x,0) = 0 \quad u_t(x,0) = 0 & x \in \mathbb{R} \end{cases}$$

define $\tilde{u}(x,t;s)$ to solve

$$\begin{cases} \tilde{u}_{tt} - c^2 \tilde{u}_{xx} = 0 & \text{in } x \in \mathbb{R} \quad t > 0 \\ \tilde{u}(x,s;s) = 0 \quad \tilde{u}_t(x,s;s) = f(x,s) & x \in \mathbb{R} \end{cases}$$

$$\tilde{u}(x,t;s) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy$$

Thm:

$$u(x,t) = \int_0^t \tilde{u}(x,t;s)\,ds = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds$$

$$= \frac{1}{2c}\int_{\Delta(x,t)} f(y,s)\,dy\,ds \quad \text{solves } (1)$$

proof:

$$u_t(x,t) = \cancelto{0}{\tilde{u}(x,t;t)} + \int_0^t \tilde{u}_t(x,t;s)\,ds$$

$$u_{tt}(x,t) = \underbrace{\tilde{u}_t(x,t;t)}_{=f(x,t)} + \int_0^t \tilde{u}_{tt}(x,t;s)\,ds$$

$$u_{xx}(x,t) = \int_0^t \tilde{u}_{xx}(x,t;s)\,ds$$

$$u_t - u_{xx} = f(x,t) + \int_0^t \overbrace{\left(\tilde{u}_{tt}(x,t;s) - \tilde{u}_{xx}(x,t;s)\right)}^{=0} ds$$

$$= f(x,t).$$

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general solution of wave eqn in 1-d

$$u(x,t) = \overset{\mathrm{I}}{\frac{1}{2}\left(\phi(x+t) + \phi(x-t)\right)} + \overset{\mathrm{II}}{\frac{1}{2}\int_{x-t}^{x+t} \psi(y)\,dy}$$

$$+ \overset{\mathrm{III}}{\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,s)\,dy\,ds}$$

picture