Heat Equation : Uniqueness in unbounded Domain (Exam Next Tuesday)

Thm Suppose $u \in C^{2,1}(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solves

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T] \\ u(x,0) = 0 & \text{in } \mathbb{R}^n \end{cases}$$

if $|u(x)| \leq A e^{ax^2}$, $A, a$ constants then

$u = 0$.

proof: Idea is to compare $u$ w/

$$f(x,t) = \frac{1}{(\delta - t)^{n/2}} e^{|x|^2/\delta - t} \quad \text{for } 0 \leq t < \delta$$

$\cancel{= \Phi(x, t-\delta)(-1)^{-n/2}}$ solves heat eqn for $0 \leq t < \delta$.

consider $v(x,t) = u(x,t) - \varepsilon f(x,t)$

then for $|x| \geq R >> 1$

$$\varepsilon f(x,t) \geq \varepsilon e^{|x|^2/\delta} > A e^{a|x|^2}$$
if $\delta \leq \frac{1}{2a}$

so then $u(x,t) - \varepsilon f(x,t) \leq 0$ on $\partial B(0,R) \times [0, \frac{\delta}{2}]$

max principle in bdd domain $\Rightarrow$

$u(x,t) - \varepsilon f(x,t) \leq 0$ in $\mathbb{R}^n \times (0, \frac{\delta}{2}]$

sending $\varepsilon \to 0$ implies $u \leq 0$ in $\mathbb{R}^n \times [0, \frac{\delta}{2}]$

repeat this argument $T/\delta$ times to

get $u(x,t) \leq 0$ in $\mathbb{R}^n \times [0,T]$.

$\square$

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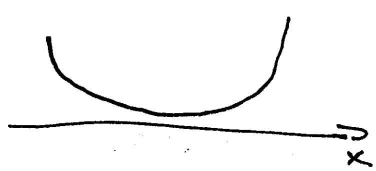
Idea (say this first):

construct a super-solution

$\partial_t f - \Delta f \geq 0$ which grows to $+\infty$

as $|x| \to \infty$



then if $u$ solves

$\partial_t u - \Delta u = 0$

$u(x,0) = 0$

and $u$ grows slower than $f$

Then $u - \varepsilon f \leq 0$ on $\partial B(0,R) \times [0,T]$

for $R$ suff large

so we can use comparison in bdd

domain then send $\varepsilon \to 0$.

e.g. $f(x,t) = M|x|^2 + 2nMt$ solves heat equation

so we could prove uniqueness for ~~and~~

solutions of heat equation growing

at most like a quadratic polynomial.