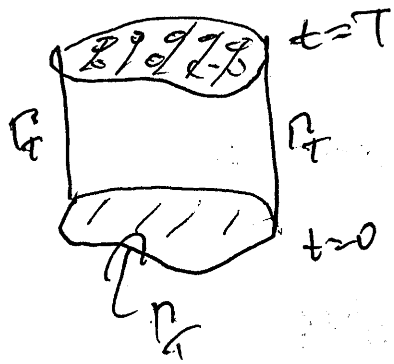
Initial / Boundary Value problems

$$(D)\quad \begin{cases} u_t - \Delta u = 0 & \text{in } U_T = U \times (0,T] \\ u(x,t) = g(x,t) & \text{on } \Gamma_T = \overline{U}_T \setminus U_T \end{cases}$$

$U \subseteq \mathbb{R}^n$ bounded <u>domain</u>



U_T called the <u>parabolic cylinder</u>

Γ_T called <u>parabolic boundary</u>

it is the sides and bottom of the cup

boundary data needs to be specified on the <u>sides</u> and <u>bottom</u> of U_T

(i.e. on Γ_T)

<u>Thm</u> ~~Suppose~~ There is at most one solution of (D) in $C^{2,1}(\overline{U}_T)$ ~~$\cap C(U_T)$~~.

<u>proof</u> Suppose u_1, u_2 both solve (D)

then $u = u_1 - u_2$ solves

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } U_T \\ u(x,t) = 0 & \text{on } \Gamma_T \end{cases}$$

multiply equation by u

$$u\partial_t u - u\Delta u = 0 \quad \text{in } V_T$$

$$\frac{\partial}{\partial t}\left(\frac{u^2}{2}\right) - u\Delta u = 0 \quad \text{in } V_T$$

integrate over V at time t.

$$0 = \int_V \frac{\partial}{\partial t}\left(\frac{u^2}{2}\right) - u\Delta u \, dx = \frac{1}{2}\frac{d}{dt}\int_V u(x,t)^2\,dx$$

$$+ \int_V |\nabla u|^2\,dx$$

$$- \int_{\partial V} u\frac{\partial u}{\partial \nu}\,dS$$

$$= \frac{d}{dt}\int_V u(x,t)^2\,dx + \int_V |\nabla u|^2\,dx$$

$$\geq \frac{d}{dt}\int_V u(x,t)^2\,dx$$

so $\frac{d}{dt}\left[\int_V u(x,t)^2\,dx\right] \leq 0$

$$\Rightarrow \int_V u(x,t)^2\,dx \leq \int_V u(x,0)^2\,dx = 0$$

$\left(u = 0 \text{ on } \Gamma_T \supseteq \bar{V} \times \{t=0\} \right.$

$$\Rightarrow \quad u(x,t) = 0 \quad \text{in } V.$$

This is called the energy method for uniqueness

Heat equation also satisfies a maximum principle

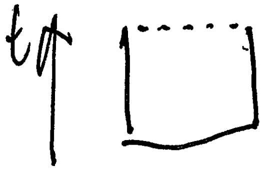
Thm (Max principle) Let $u \in C(\overline{U_T}) \cap C^{2,1}(\overline{U_T})$

solve $\left| \, u_t = \Delta u \quad \text{in } U_T , \right.$

then $$\max_{\overline{U_T}} u(x,t) = \max_{\Gamma_T} u(x,t) .$$

i.e. maximum in parabolic domain

occurs on parabolic boundary



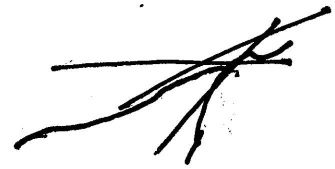
parabolic boundary is sides and bottom of the cup.

proof: First suppose $\begin{cases} \partial_t u - \Delta u < 0 & \text{in } U_T \\ \cancel{}\ u < \overset{0}{\cancel{\max_{\overline{U_T}} u}} & \text{on } \Gamma_T \\ \cancel{\text{at } t=0} & \end{cases}$

then by continuity $u(x,t) < \overset{0}{\cancel{\max_{\overline{U_T}} u}}$ for t sufficiently small.

Let t_0 to be the first time that

$$\max_{x \in V} u(x,t) \geq 0$$



$$t_0 = \inf \{ t \in (0,T] : \exists\, x_0 \in V \text{ w/ } u(x_0,t) \geq 0 \}$$

if the set being infimized over is empty we are done. If not then

$\exists$ a sequence $x_n \in V$ and $t_n \downarrow t_0$
(WLOG $x_n \to x_0 \in \bar{V}$ as $n \to \infty$)

so that $u(x_n, t_n) \geq 0$

since $u < 0$ on Γ_T and $u \in C(\bar{V}_T)$

$\Rightarrow x_0 \in V$

so we have $(x_0, t_0) \in V_T$

so that $u(x_0, t_0) = 0$

$u(x,t) \leq 0$ for $x \in \bar{V}$, $t \leq t_0$

($u(x,t) < 0$ for $x \in \bar{V}$, $t < t_0$)

$\Rightarrow$ x_0 is interior global max of $u(\cdot, t_0)$

so $\Delta u(x_0, t_0) \leq 0$

and

$$\partial_t u(x_0,t_0) = \lim_{h\to 0} \frac{u(x_0,t) - u(x_0,t_0-h)}{h} \geq 0$$

so $\left(\partial_t u - \Delta u\right)(x_0,t_0) \geq 0$

Which is a contradiction.

Now we need to do general case

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U_T \end{cases}$$

Consider ~~let~~ $v_\varepsilon(x,t) = u(x,t) - \max_{\Gamma_T} u - \varepsilon - \varepsilon t$

$v_\varepsilon \leq 0 - \varepsilon - \varepsilon t < 0$ on Γ_T

and $\partial_t v_\varepsilon - \Delta v_\varepsilon = \partial_t u - \Delta u - \varepsilon$

$= -\varepsilon < 0$

so our previous argument applies to get

$v_\varepsilon(x,t) \leq 0$ in U_T or

$u(x,t) \leq \max_{\Gamma_T} u + \varepsilon + \varepsilon t$ in U_T

send $\varepsilon \to 0$. $\square$

Remark: You should recognize the method from Laplace equation. prove max principle for strict subsolutions then make a perturbation. For strict subsolutions use the conditions at interior local max to contradict the equation.