~~Simp~~ Random Walks, Brownian Motion and Heat/Laplace Eqn

Consider a simple random walk on $\mathbb{Z}$

$(X_k)_{k\in\mathbb{N}}$

X_k

$$X_{k+1} = \begin{cases} X_k + 1 & \text{with prob } \frac{1}{2} \\ X_k - 1 & \text{w/ prob } \frac{1}{2} \end{cases}$$

independent coin flips at each step.

$X_0 = 0$

for a different initial point $x \in \mathbb{Z}$

call X_k^x SRW started at x.

Can also choose initial point x randomly

from a distribution $\rho : \mathbb{Z} \to \mathbb{R}_+$

$$\sum_{x\in\mathbb{Z}} \rho(x) = 1.$$

Call this $(X_k^\rho)_{k\in\mathbb{N}\cup\{0\}}$

What is the probability distribution function for the location of X_k^ρ?

Call $u(x,k) = \mathbb{P}(X_k^\rho = x)$

Then $u(x,k+1) = \mathbb{P}(X^{\rho}_{k+1} = x)$

$$= \frac{1}{2}\mathbb{P}(X^{\rho}_{k} = x+1) + \frac{1}{2}\mathbb{P}(X^{\rho}_{k} = x-1)$$

$$= u(x,k) + \frac{1}{2}\left(u(x+1,k) + u(x-1,k) - 2u(x,k)\right)$$

or rearranging

$$\boxed{u(x,k+1) - u(x,k) = \frac{1}{2}\left(u(x+1,k) + u(x-1,k) - 2u(x,k)\right)}$$

discrete heat equation.

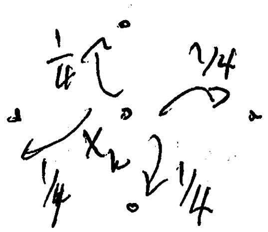
discrete heat equation tracks the evolution of the probability distribution function for Simple Random Walk

cts heat equation tracks evolution of pdf for Brownian motion.

Laplace Equation;

Consider again the SRW now on $\mathbb{Z}^2$ (just to see how Laplace comes up instead of another PDE operator consistent w/ Laplace in 1-D).

let $\Lambda \subseteq \mathbb{Z}^2$ a connected, bounded region.



SRW

Call $\partial_{out}\Lambda$ the outer vertex boundary, $x \in \mathbb{Z}^2$ so that one of $x \pm e_1, x \pm e_2$ is in Λ

and let $g : \partial_{out}\Lambda \to \mathbb{R}$.

look at $u(x) = \mathbb{E}\left(g\left(X^x_{\tau_x(\Lambda)}\right)\right)$

where $\tau_x(\Lambda) = \inf\{k \geq 0 : X^x_k \notin \Lambda\}$.

note $\tau_x(\Lambda)$ is a random variable

$$X^x_{\tau_x(\Lambda)} \in \partial_{out}\Lambda$$

~~Think of u as~~

$$u(x) = \frac{1}{4}\left(u(x+e_1) + u(x-e_1) + u(x+e_2) + u(x-e_2)\right)$$

or

$$0 = \frac{1}{4}\underbrace{\left[\left(u(x+e_1) + u(x-e_1) - 2u(x)\right) + \left(u(x+e_2) + u(x-e_2) - 2u(x)\right)\right]}_{\text{discrete Laplace operator on } \mathbb{Z}^2}.$$

So
$$\begin{cases} \Delta_{\mathbb{Z}^2} u(x) = 0 & x \in \Lambda \\ u(x) = g(x) & x \in \partial_{out}\Lambda \end{cases}$$

a Dirichlet problem for Laplace Operator.

So solution of Dirichlet problem can be interpreted as the expected value of g at the location where Brownian motion started at x leaves domain U.

~~Numerics~~

Finite Difference Schemes

The Heat Equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = f(x) & \text{in } \mathbb{R}^n \end{cases}$$

- Smoothing
- Energy dissipation (Entropy dissipation)
- Backwards ill-posedness
- maximum principle
-

Mass Conservation if u solves heat equation w/ sufficient decay

$$\frac{d}{dt}\int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n} u_t(x,t)\,dx = \int_{\mathbb{R}^n} \Delta u(x,t)\,dx$$

$$= 0$$

so $$\int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n} u(x,0)\,dx$$

Scaling Invariance

suppose we have u solving

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^n$$

when is $u_\lambda(x,t) = u(\lambda x, \lambda^a t)$ a solution for all $\lambda > 0$?

$$\partial_t u_\lambda = \lambda^a \partial_t u$$
$$\Delta u_\lambda = \lambda^2 \Delta u$$

so $\partial_t u_\lambda - \Delta u_\lambda = \lambda^a \partial_t u - \lambda^2 \Delta u$
$= 0$
when $a = 2$.

so $u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$ is a solution.

Let's look for a scale invariant solution ~~on R~~ on $\mathbb{R}\times[0,\infty)$

$$\Phi(x,t) = \lambda^{\alpha}\Phi_\lambda(x,t) = \lambda^{\alpha}\Phi(\lambda x, \lambda^2 t) \quad \forall \lambda>0.$$

to preserve mass under the rescaling

$\alpha = 1$

note taking $\lambda = t^{-1/2}$

$$\Phi(x,t) = t^{-\alpha/2}\Phi\left(\frac{x}{\sqrt{t}}, 1\right)$$

so Φ is determined just by a function of $y = \frac{x}{\sqrt{t}}$ $\quad v(y) = \Phi(y,1)$

$$\Phi(x,t) = \frac{1}{t^{\alpha/2}} v\left(\frac{x}{\sqrt{t}}\right)$$ again, since

solutions of heat eqn have $\frac{d}{dt}\int u = 0$

need to choose $\alpha = 1$, ~~but~~

~~let's just wait and get this requirement~~

~~from the equation~~

$$0 = \partial_t\Phi - \Phi_{xx} = -\frac{\alpha}{2}\frac{1}{t^{1+\alpha/2}}v - \frac{1}{2}\frac{1}{t^{\alpha/2}}\frac{x}{t^{3/2}}v' - \frac{1}{t^{1+\alpha/2}}v''$$

$$= -\frac{1}{t^{1+\alpha/2}}\left(v'' + \frac{1}{2}\frac{x}{\sqrt{t}}v' + \frac{\alpha}{2}v\right)$$

with $\alpha = 1$

$$v'' + \frac{1}{2} y v' + \frac{1}{2} v = 0$$

$$v'' + \frac{1}{2} (yv)' = 0$$ integrating

$\cancel{v'' + \frac{1}{2}}$

$$v' + \frac{1}{2} y v = A$$ using an integrating factor

$$\left(e^{y^2/4} v\right)' = A e^{y^2/4}$$

we can set $A = 0$ since we are just looking for one solution

$$e^{y^2/4} v(y) = B \quad \text{so} \quad v(y) = B e^{-y^2/4}$$

We choose B so that

$$1 = \int_{-\infty}^{\infty} v(y)\, dy = B \int_{\mathbb{R}} e^{-y^2/4}\, dy = B\sqrt{4\pi}$$

so $$v(y) = \frac{1}{\sqrt{4\pi}} e^{-y^2/4}$$

$$\boxed{\Phi(x,t) = \frac{1}{t^{1/2}} v\left(\frac{x}{t^{1/2}}\right) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}}$$ the fundamental solution of heat eqn

$$\begin{cases} \Phi_t - \Phi_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ \Phi(x,0) = ? \end{cases}$$

We will see $\Phi(x,0) = \delta_0$

<u>Higher dimensions</u>

In $\mathbb{R}^n$ $\Phi(x,t) = \Phi_1(x_1,t) \cdots \Phi_1(x_n,t)$

~~$\Phi_n(x) = \Phi_1(x_1) \cdots \Phi_1(x_n)$~~

where $\Phi_1 : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ is the 1-d fundamental solution.

$$\partial_t \Phi - \Delta \Phi = \sum_{j=1}^n \partial_t \Phi_1(x_j,t) \prod_{i \neq j} \Phi(x_i,t)$$

$$- \sum_{j=1}^n \Phi_{1,xx}(x_j,t) \prod_{i \neq j} \Phi(x_i,t)$$

$$= 0 .$$

so $\Phi(x,t) = \dfrac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}$

Properties of the fundamental Solution

$\Phi(x,t)$ solves heat equation away from $(0,0)$, so

$\Phi(x-y,t)$ solves away from $(y,0)$

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy$$

$$= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t}\, g(y)\, dy$$

should be a solution as well

Thm Let $g \in C(\mathbb{R}^n)$ be bounded and ~~$u(x,t) = \Phi_t * g$~~

$u(x,t) = \left(\Phi(\cdot,t) * g\right)(x)$ Then

1. $u(\cdot,t) \in C^\infty(\mathbb{R}^n)$ for every $t > 0$
2. $u_t - \Delta u = 0$ for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$
3. $\lim_{(x,t) \to (x_0,0^+)} u(x,t) = g(x_0)$

proof: $\Phi(x,t)$ is C^∞ and all derivatives have exponential decay so differentiating under integral will be justifiable.

$$u_t - \Delta u = \int_{\mathbb{R}^n} \left(\Phi_t(x-y,t) - \Delta \Phi(x-y,t) \right) g(y)\,dy$$

$$= 0$$

so u solves heat eqn

Let $\varepsilon > 0$, since $g \in C(\mathbb{R}^n)$ $\exists$ $\delta > 0$ so that $|x - x^0| \le \delta \Rightarrow |g(x) - g(x^0)| \le \varepsilon$

$$|u(x,t) - g(x^0)| = \left| \int_{\mathbb{R}^n} \Phi(x-y,t)(g(y) - g(x^0))\,dy \right|$$

(by usual trick since $\int_{\mathbb{R}^n} \Phi(x-y)\,dy = 1$)

$$\le \int_{\mathbb{R}^n} \Phi(x-y,t)\,|g(y) - g(x^0)|\,dy$$

now for $|y - x_0| \le \delta$ $|g(y) - g(x_0^{\circ})| \le \varepsilon$ while for $|y - x_0| \ge \delta$ $\Phi(x-y,t)$ will have vanishing mass as $t \to 0$

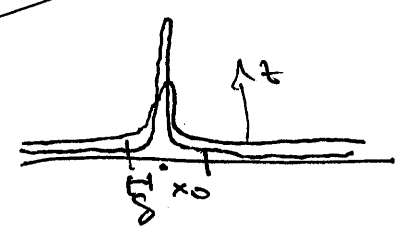
that ~~Suppose $|x - x_0| \le \frac{\delta}{2}$~~

though

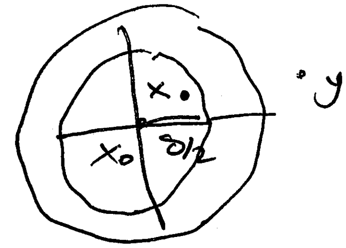
$$|u(x,t) - g(x^0)| \le \int_{|y-x^0|\le\delta} \Phi(x-y,t)\,|g(y) - g(x^0)|\,dy$$

$$+ \int_{|y-x^0|\ge\delta} \Phi(x-y,t)\,|g(y) - g(x^0)|\,dy$$

$$\le \varepsilon \cdot \underbrace{\int_{|y-x_0|\le\delta} \Phi(x-y,t)\,dy}_{\le 1} + 2\left(\sup_{\mathbb{R}^n}|g|\right) \int_{|y-x^0|\ge\delta} \Phi(x-y,t)\,dy$$



$$= \int_{|x^0-y|\ge\delta} \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/4t}\,dy$$



now if $|x - x^0| \le \frac{\delta}{2}$

$\Rightarrow |y - x| \ge \frac{\delta}{2} \ge \frac{1}{2}|y - x^0|$

for y in the region of integration

$$\le \int_{|x^0-y|\ge\delta} \frac{1}{(4\pi t)^{n/2}} e^{-|x^0-y|^2/16t}\,dy$$

changing variables to

$z = \frac{y-x_0}{4\sqrt{t}}$ $\quad dz = \frac{1}{(4\sqrt{t})^n} dy$

$$= \int_{|z| \geq \frac{\delta}{4\sqrt{t}}} \frac{(16t)^{n/2}}{(4\pi t)^{n/2}} e^{-|z|^2} dz$$

$$= \left(\frac{4}{\pi}\right)^{n/2} \int_{|z| \geq \frac{\delta}{4\sqrt{t}}} e^{-|z|^2} dz \to 0 \text{ as } t \to 0$$

since $e^{-|z|^2}$ is integrable. ∎

This is a (rigorous) justification that $\Phi(\cdot, t) \to \delta_0$ in sense of distributions as $t \to 0$.

~~Energy Method~~

Duhamel's Principle for the Inhomogeneous Heat Eqn

$$\begin{cases} u_t - \Delta u = f(x,t) & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) \end{cases}$$

~~we can think of f~~

we will build solution again using Φ

note $\Phi(x-y, t-s)$ solves heat

eqn in $\mathbb{R}^n \times (s, \infty)$

and $\lim_{t \to s^+} \Phi(\cdot - y, t-s) = \delta_y(\cdot)$

we define $u(x,t;s)$ be

which is in a sense the influence

first of the forcing term f at time s

on future times

$$\begin{cases} \partial_t u(x,t;s) - \Delta u(x,t;s) = 0 & \text{in } \mathbb{R}^n \times (s,\infty) \\ u(x;s;s) = f(x,s) & \text{in } \mathbb{R}^n \end{cases}$$

so we already know a solution

$$u(x,t;s) = \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s) \, ds$$

Then Duhamel's principle says

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y) dy + \int_0^t u(x,t;s) \, ds$$

$$= \int_{\mathbb{R}^n} \Phi(x-y,t) g(y) dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s) dy \, ds$$

Thm: let $f \in C^{2,1}(\mathbb{R}^n \times (0,\infty))$ and $g \in C(\mathbb{R}^n)$ bdd then $u(x,t)$ from Duhamel's formula solves

(2) $\begin{cases} u_t - \Delta u = f(x,t) & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) & \text{in } \mathbb{R}^n \end{cases}$

(1) $u \in C^{2,1}(\mathbb{R}^n \times (0,\infty))$

(3) $\lim_{\substack{(x,t) \to (x_0,0) \\ t>0}} u(x,t) = g(x)$.

<u>proof</u> we just need to analyse

$$v(x,t) = \int_0^t u(x,t;s)\,ds$$

check that it is regular and solves

$$\begin{cases} v_t - \Delta v = f(x,t) & \text{in } \mathbb{R}^n \times (0,\infty) \\ v(x,0) = 0 & \text{in } \mathbb{R}^n \end{cases}$$

Note: by previous result $u(x,t;s)$ is ~~$v_t(x,t) =$~~ ~~$u(x,s;s)$~~ cts as $t \to s$

$u(x,t;s) \to f(x,s)$ ~~ptwise~~ unif in $\mathbb{R}^n$ since $f \in C_c(\mathbb{R}^n)$

$$v_t(x,t) = u(x,t;t) + \int_0^t u_t(x,t;s)\,ds$$

$$= f(x,t) + \int_0^t u_t(x,t;s)\,ds$$

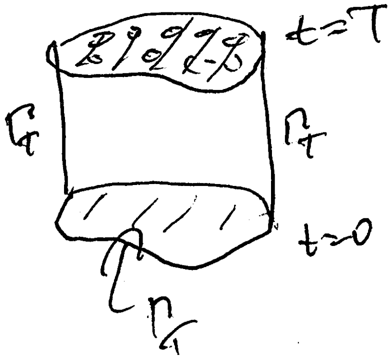
$$\Delta v(x,t) = \int_0^t \Delta u(x,t;s)\,ds$$

$$v_t - \Delta v = f(x,t) + \int_0^t \left(u_t(x,t;s) - \Delta u(x,t;s)\right) ds$$

$$= f(x,t) \,.$$

Initial / Boundary Value problems

$$(D) \begin{cases} u_t - \Delta u = 0 & \text{in } U_T = U \times (0,T] \\ u(x,t) = g(x,t) & \text{on } \Gamma_T = \overline{U}_T \setminus U_T \end{cases}$$



U_T called the <u>parabolic cylinder</u>

Γ_T called <u>parabolic boundary</u>

it is the sides and bottom of the cup

boundary data needs to be specified on the <u>sides</u> and <u>bottom</u> of U_T

(i.e. on Γ_T)

<u>Thm</u> ~~Suppose~~ There is at most one solution of (D) in $C^{2,1}(\overline{U}_T)$ $\cancel{\cap C(U_T)}$.

<u>proof</u>