Green's function of $B(0,1)$

$$\tilde{x} = \frac{x}{|x|^2} \quad , \quad |\tilde{x}| = \frac{1}{|x|}$$

$$G(x,y) = \Phi(x-y) - \Phi(|x|(\tilde{x}-y))$$

for $x \in B(0,1)$, $y \in \partial B(0,1)$ want to show

$G(x,y) = 0$, need to show

$$|x-y| = |x||\tilde{x}-y|$$

$$|x|^2|\tilde{x}-y|^2 = \left| |x| \cdot \frac{x}{|x|^2} - |x|y \right|^2 = \left| \frac{x}{|x|} - |x|y \right|^2$$

$$(|y|=1) \qquad = 1 - 2y\cdot x + |x|^2$$

$$= |x-y|^2$$

So $G(x,y)$ has all the necessary properties.

Solution of
$$\begin{cases} -\Delta u = \cancel{f}\, 0 & \text{in } B_1 \\ u = g & \text{on } \partial B_1 \end{cases}$$

$$u(x) = \int_{\partial B(0,1)} -\frac{\partial G}{\partial \nu_y}(x,y)\, g(y)\, dS_y$$

$K(x,y) = -\frac{\partial G}{\partial \nu_y}(x,y)$ ~~it~~ is called the Poisson Kernel of the unit ball.

$$\frac{\partial G}{\partial y_i} = \frac{\partial \Phi}{\partial y_i}(x-y) - \frac{\partial}{\partial y_i}\Phi(|x|(\tilde{x}-y))$$

$$= \frac{1}{n\alpha(n)}\frac{x_i - y_i}{|x-y|^n} - \frac{|x|^2(\tilde{x}_i - y_i)}{(|x||\tilde{x}-y|)^n}\frac{1}{n\alpha(n)}$$

$$= \frac{1}{n\alpha(n)}\frac{x_i - y_i}{|x-y|^n} - \frac{1}{n\alpha(n)}\frac{x_i - |x|^2 y_i}{|x-y|^n}$$

$$= -\frac{1}{n\alpha(n)}\frac{1-|x|^2}{|x-y|^n}\, y_i$$

then
$$-\frac{\partial G}{\partial \nu}(x,y) = -\sum_i y_i \frac{\partial G}{\partial y_i} = \frac{+1}{n\alpha(n)}\frac{1-|x|^2}{|x-y|^n}\sum_i y_i^2$$

$$= \frac{1}{n\alpha(n)}\frac{1-|x|^2}{|x-y|^n}$$

$$\boxed{K(x,y) = \frac{1}{n\alpha(n)}\frac{1-|x|^2}{|x-y|^n}}$$ Poisson Kernel of unit ball

~~Energy Methods~~

Thm: Assume $g \in C(\partial B(0,r))$ and define u by

$$u(x) = \int_{\partial B(0,r)} \frac{1}{n\alpha(n) r} \frac{r^2 - |x|^2}{|x-y|^n} g(y)\, dS_y$$

then (i) $u \in C^\infty(B(0,r))$

(ii) $\Delta u = 0$ in $B(0,r)$

(iii) $\lim_{\substack{x \to x_0 \in \partial B(0,r) \\ x \in B(0,r)}} u(x) = g(x_0)$

Energy Methods

Previous ~~approach~~ results have relied on representation formula mostly (not all). An alternative perspective on Laplace's equation comes from the energy interpretation

Consider

$$(D)\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

Thm There exists at most one solution $u \in C^2(\bar{U})$ of (D).

proof: Suppose v solves (D) as well.

set $w(x) = u(x) - v(x)$.

$$\begin{cases} -\Delta w = 0 & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}$$

multiply by w and integrate

$$0 = \int_U -w\Delta w \, dx = \int_U |\nabla w|^2 \, dx \mp \int_{\partial U} w \frac{\partial w}{\partial \nu} \, dS$$

$$= \int_U |\nabla w|^2 \, dx$$

Thus $|\nabla w| = 0$ in U

since $w = 0$ on ∂U

$w = 0$ in U.

Thm (Maximum Principle by Energy Method) $-\Delta u = 0$ in U then

$$\max_{U} u = \max_{\partial U} u$$

proof: call $w = (u - \max_{\partial U} u)_+$

(not C^2 but let's pretend)

$w = 0$ on ∂U

$$0 = \int_U -w \Delta u \, dx = \int_U \nabla w \cdot \nabla u \, dx - \int_{\partial U} w \frac{\partial u}{\partial \nu} \, dS$$

$$= \int_U \nabla w \cdot \nabla u \, dx$$

$$\nabla w = \begin{cases} \nabla u(x) & \text{if } u(x) > \max_{\partial U} u \\ 0 & \text{else} \end{cases}$$

$$= \int_{u > \max_{\partial U} u} |\nabla u|^2 \, dS$$

so $\nabla u = 0$ on $\{x \in U : u > \max_{\partial U} u\}$

and $u = \max_{\partial U} u$ on $\partial \{x \in U : u > \max_{\partial U} u\}$ $\subset\subset U$

by continuity of u so $u \equiv \max_{\partial U} u$ in to make this argument rigorous

let $\phi : \mathbb{R} \to [0,\infty)$ smooth approximation

of ~~$H(x)=$~~ $M(x) = \max\{x, 0\}$

$\phi' > 0$ when $x > 0$

$\phi = \phi' = 0$ for $x < 0$

call $w(x) = \phi(u(x) - \max_{\partial V} u)$

$\nabla w = \phi'(u - \max_{\partial V} u)\, \nabla u(x)$

Rest of the argument will be the same.

Why does this sort of argument work?

$$\begin{cases} -\Delta u = f & \text{in } V \\ u = g & \text{on } \partial V \end{cases}$$

is the Euler-Lagrange equation for

an energy minimization.

(i.e. it is like the first derivative test)
possibly w/ Lagrange multiplier

We define the ~~Dirichlet~~ Energy Functional

$$I[w] := \int_U \frac{1}{2}|Dw|^2 - wf \, dx$$

(called Dirichlet energy when $f = 0$.)

defined on the admissible set

$$\mathcal{A} = \{ w \in C^2(\bar{U}) : w = g \text{ on } \partial U \}$$

Thm (Dirichlet's Principle) Assume $u \in C^2(\bar{U})$ solves (D)

then

$$I[u] = \min_{w \in \mathcal{A}} I[w] \qquad (M)$$

Conversely if $u \in \mathcal{A}$ satisfies (M) then u solves (D).

Rm: We still have not justified that there exists u solving (D) or u minimizing (M) proof: one direction is a homework problem.

lets show the other direction:

if $I[u] = \min_{w \in \mathcal{A}} I[w]$ then

u solves (D).

the boundary condition is already imbedded in the definition of $\mathcal{A}$, just need to check $\boxed{-\Delta u = f \text{ in } U}$

let $\varphi \in C_c^\infty(U)$

$u + \varepsilon\varphi \in \mathcal{A}$ so

$\gamma(\varepsilon) := I[u+\varepsilon\varphi]$ has a minimum at $\varepsilon = 0$

(think of  moving along a path in $\mathcal{A}$ in direction φ)

so

$$0 = \gamma'(0) = \int_U \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{1}{2}|\nabla u + \varepsilon \nabla\varphi|^2 - \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(u+\varepsilon\varphi)\, dx$$

$$0 = \int_V \nabla u \cdot \nabla \varphi - f\varphi \, dx$$

$$= \int_{\partial V} \varphi \frac{\partial u}{\partial \nu} \, dS + \int_V (-\Delta u)\varphi - f\varphi \, dx$$

$\varphi = 0$ on ∂V so

$$\int_V (-\Delta u - f)\varphi \, dx = 0 \quad \text{for all } \varphi \in C_c^\infty(V)$$

$\Rightarrow$ $-\Delta u = f$ ptwise in V.

Existence:

So far we have only pointed how to prove existence for Dirichlet / Neumann problems in $B(0,1)$ and $\mathbb{R}^n_+$ upper half space.

What about in general domains? There are several methods let me point to two.

- Minimize $I[w]$ over $\mathcal{A}$, take a ~~minimi~~ infimizing sequence and show it has a convergent subsequence.

(this is not so easy minimizing sequence ~~may~~ will not actually converge in $C^2(\bar{U})$ in general.)

- Perron's Method (or the method of sub/super solutions)

Let $v \in C(\bar{U})$ be <u>sub-harmonic</u>

in the sense that

$$v(x) \le \fint_{B(x,r)} v(y)\,dy \quad \forall\ B(x,r) \subseteq U$$

if u solves $\begin{cases} -\Delta u = 0 & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$

~~the~~ and v subharmonic in U

w/ $v \le g$ on ∂U

then $v \le u$ in U.

it turns out that we can characterize u

by

$$u(x) := \sup\{v(x) : v \text{ subsolution of } (D)\}$$

Probabalistic Interpretation of Dirichlet problem

Consider a simple random walk (SRW)

In $Q_N := \{-N, \dots, N\}^{2} \subseteq \mathbb{Z}^{2}$.

$$X_0^{x_0} = x_0 \in Q_N$$

$$\cancel{X_{k+1} = X_k \pm e_j \quad \text{w.p.} \quad \frac{1}{2d}}$$

$$X_{k+1}^{x_0} = \begin{cases} X_k + e_1 & \text{with prob } \frac{1}{2} \\ X_k - e_1 & \text{" " } \frac{1}{2} \\ X_k + e_2 & \text{" " } \frac{1}{2} \\ X_k - e_2 & \text{" " } \frac{1}{2} \end{cases}$$

~~Let $E \subseteq \partial_{out} Q_N$~~

~~Consider $u : Q_N \to \mathbb{R}$~~

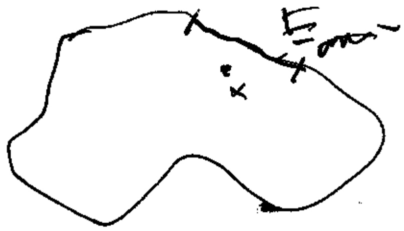
$$\cancel{u(x) = \mathbb{P}(X^x}$$

Call $T_x = \min_{k>0} \{ k > 0 : X_k^x \notin Q_N \}$

the exit time of the SRW from Q_N

Consider $E \subseteq \partial_{out} Q_N = \{ x \in \mathbb{Z}^2 : x \in Q_N^c \text{ but } x \pm e_j \in Q_N \text{ for some } \pm, j \}$

What is the probability that the SRW started from x leaves Q_N for the first time through E



E ~ target set

$$u(x) := \mathbb{P}(X^x_{\tau_x} \in E)$$

$$u(x) = \frac{1}{4}\Big(\overbrace{\mathbb{P}(X^{x+e_1}_{\tau_{x+e_1}} \in E)}^{= u(x+e_1)} + \overbrace{\mathbb{P}(X^{x-e_1}_{\tau_{x-e_1}} \in E)}^{= u(x-e_1)} + u(x+e_2) + u(x-e_2)\Big)$$

$$0 = \frac{1}{4} \sum_{|y-x|=1}' (u(y) - u(x)) =: \Delta u(x)$$

$\uparrow$ discrete Laplacian.

So u is a discrete <u>harmonic</u> function

w/ $u(x) = 1$ on E

$u(x) = 0$ on $\partial_{out} Q_N \setminus E$.

i.e. it solves a Dirichlet problem

$$\begin{cases} -\Delta u(x) = 0 & x \in Q_N \\ u(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases} & \text{for } x \in \partial_{?} Q_N \end{cases}$$

Finite Difference Scheme:

This ~~also gives an idea for~~ is related to a very natural numerical approximation scheme for continuum Dirichlet problems, called

a finite difference scheme

approximate

$\partial_{x_i} u$ by $\dfrac{u(x+he_i) - u(x)}{h} =: D_h^{+i} u(x)$

$\partial_{x_i}^2 u$ by $D_h^{+i} D_h^{-i} u(x) = \dfrac{u(x+he_i) + u(x-he_i) - 2u(x)}{h^2}$

and $\Delta_h u(x) = \dfrac{1}{h^2} \sum_i^{d} [u(x+he_i) + u(x-he_i) - 2u(x)]$

~~Note this $\Delta_h u(x)$ is like $\frac{4}{h^2} \Delta_{disc}$ on~~

~~$\mathbb{Z}^d$ on $h\mathbb{Z}^d$.~~

Laplace Eqn and Brownian Motion

In a similar vein to the SRW description

(since Brownian motion is scaling limit of random walk)

can view
$$\begin{cases} -\Delta u = 0 & \text{in } V \\ u = g & \text{on } \partial V \end{cases}$$

as $u(x) = \mathbb{E}_x\, g(B^x_{\tau_x})$

expected value of g at the location when B^x_t exits V.