Green's fcn Boundary Data

Thm: Let $u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n} dy \quad x \in \mathbb{R}^n_+$

for $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$. Then

(i) $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$

(ii) $-\Delta u = 0 \quad$ in $\mathbb{R}^n_+$

(iii) $\lim_{\substack{x \to x_0 \in \partial\mathbb{R}^n_+ \\ x \in \mathbb{R}^n_+}} u(x) = g(x_0)$

proof: 1. for fixed x $\exists$ $y \mapsto G(x,y)$ is harmonic except for at $y = x$.

As G is symmetric
$x \mapsto G(x,y)$ is harmonic except for $x=y$
$x \mapsto -\frac{\partial G}{\partial y_n}(x,y)$ " " " "
for $y \in \partial\mathbb{R}^n_+$, $x \in \mathbb{R}^n_+$ $x \neq y$
so $-\frac{\partial G}{\partial y_n}(x,y)$ harmonic

2. $\int_{\partial\mathbb{R}^n_+} K(x,y)\, dy = 1$

3. Fix $x_0 \in \partial\mathbb{R}^n_+$, $\varepsilon > 0$. Choose δ small s.t. $|g(y) - g(x_0)| \le \varepsilon$ if $|y - x_0| \le \delta$.

Then if $|x - x_0| \le \frac{\delta}{2}$, $x \in \mathbb{R}^n_+$

$$|u(x) - g(x_0)| = \left|\int_{\partial\mathbb{R}^n_+} K(x,y)\, g(y)\, dy - g(x_0)\right|$$

$$= \left|\int_{\partial\mathbb{R}^n_+} K(x,y)\big(g(y) - g(x_0)\big)\, dy\right|$$

$$\leq \int_{\partial\mathbb{R}^n_+ \setminus B(x_0,\delta)} K(x,y)\,|g(y) - g(x_0)|\, dy$$

$$+ \int_{\partial\mathbb{R}^n_+ \cap B(x_0,\delta)} K(x,y)\,|g(y) - g(x_0)|\, dy$$

$$= I + J$$

$$J \leq \varepsilon \int_{\partial\mathbb{R}^n_+ \cap B(x_0,\delta)} K(x,y)\, dy \leq \varepsilon$$

for $|x - x_0| \leq \frac{\delta}{2}$, and $|y - x_0| \geq \delta$

$$|y - x| \geq |y - x_0| - |x - x_0| \geq |y - x_0| - \frac{\delta}{2}$$

$$\geq \frac{1}{2}|y - x_0|$$

so

$$I \leq 2\|g\|_\infty \int_{\partial\mathbb{R}^n_+ \setminus B(x_0,\delta)} K(x,y)\, dy$$

$$\leq \frac{4\|g\|_\infty x_n}{n\,\alpha(n)} \int_{\partial\mathbb{R}^n_+ \setminus B(x_0,\delta)} \frac{2^n}{|y - x_0|^n}\, dy$$

$$\to 0 \quad \text{as} \quad x_n \to 0.$$