Boundary Value Problems

$U \subseteq \mathbb{R}^n$ open and bounded

$$(D) \quad \begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases} \qquad (N) \quad \begin{cases} -\Delta u = f & \text{in } U \\ \frac{\partial u}{\partial \nu} = h & \text{on } \partial U \end{cases}$$

Dirichlet　　　　　　　　　　Neumann

$f \in C(U)$, $\quad g, h \in C(\partial U)$

Thm There is at most one solution $u \in C^2(U) \cap C(\overline{U})$ of (D) (or (N)) :
up to adding a constant

proof: let u_1, u_2 be solutions of (D)

then $w = u_1 - u_2$ solves

$$\begin{cases} -\Delta w = 0 & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}$$

max principle $\Rightarrow$ $w = 0$

proof 2: ~~$0 = w$~~ multiply by w and integrate

$$0 = \int_U w \Delta w \, dx = \int_{\partial U} w \frac{\partial w}{\partial \nu} \, dS - \int_U |Dw|^2 \, dx$$

since $w=0$ on ∂U

$$\int_U |\nabla w|^2 \, dx = 0$$

(same proof works when $\frac{\partial w}{\partial \nu} = 0$ on ∂U)

<u>Note</u>: for maximum principle uniqueness proof of Neumann problem w/ $w \in C^2(U) \cap C^1(\bar{U})$

$$\begin{cases} -\Delta w = 0 & \text{in } U \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$$

$\Rightarrow$ ~~$w=0$~~. $w = c$ constant in U (if U is connected).

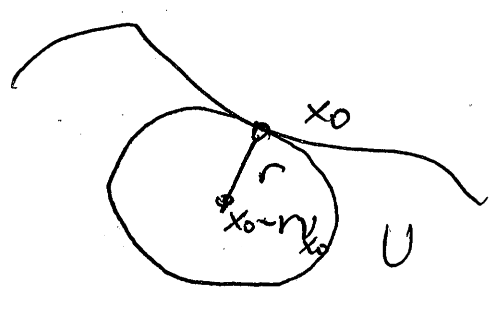
If w non-constant strong maximum principle implies $\max_{\partial U} w > w(x)$ for all $x \in U$.

$\Rightarrow$ $\exists$ $x_0 \in \partial U$ w/ $w(x_0) > w(x)$ for all $x \in U$.

<u>Lemma</u> (Hopf maximum principle) w/ interior ball property
If w harmonic in U, ~~and has strict~~

$x_0 \in \partial U$ s.t. $w(x_0) = 0$ and $w(x) > 0$ in U

then $\frac{\partial w}{\partial \nu}(x_0) < 0$.



Harmonic Regularity

if $u \in C^2(U)$ is harmonic

$$-\Delta u = 0 \quad \text{in } U$$

then automatically u is C^∞

the PDE imposes extra regularity

Thm If $u \in C(U)$ satisfies MVP ~~then~~ for each ball $B(x,r) \subseteq U$ then

$$u \in C^\infty(U).$$

proof: let ρ be a standard mollifier

$$\rho(x) = \tilde{\rho}(|x|) = \begin{cases} e^{-\frac{1}{1-|x|^2}} \cdot C & |x| \le 1 \\ 0 & |x| \ge 1 \end{cases}$$

so that $\int_{\mathbb{R}^n} \rho(x)\, dx = 1$.

~~As~~ call $\rho_\varepsilon = \varepsilon^{-n} \rho\left(\frac{x}{\varepsilon}\right)$. ~~As you~~

~~will see on homework~~

Set $u^\varepsilon = \rho_\varepsilon * u$ in $U_\varepsilon = \{x \in U : dist(x, \partial U) > \varepsilon\}$

$$u^\varepsilon(x) = \int_{\mathbb{R}^n} \rho_\varepsilon(x-y)\, u(y)\, dy$$

Well defined ~~smooth~~ in U^ε since

$\varepsilon^{-n} \rho\left(\frac{x-\cdot}{\varepsilon}\right)$ supported in $\frac{|x-y|}{\varepsilon} \le 1$

$|x-y| \le \varepsilon$ $\quad x \in U_\varepsilon$ $\Rightarrow$ $y \in U$.

u^ε is actually in $C^\infty(U_\varepsilon)$ since

$$D^k u^\varepsilon(x) = \int_{\mathbb{R}^n} D^k \rho_\varepsilon(x-y)\, u(y)\, dy$$

(actual justification using difference quotients etc)

We will prove that actually $u^\varepsilon \equiv u$ in U_ε and hence u is smooth as well

$$u^\varepsilon(x) = \int_U \rho_\varepsilon(x-y)\, u(y)\, dy$$

$$= \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \rho_\varepsilon(x-y)\, u(y)\, dy$$

$$= \frac{1}{\varepsilon^n} \int_0^\varepsilon \rho^{rad}\left(\frac{r}{\varepsilon}\right) \left[\int_{\partial B(x,r)} u(y)\, dS_y\right] dr$$

(MVP) $$= \frac{1}{\varepsilon^n} \int_0^\varepsilon r^{n-1} \rho^{rad}\left(\frac{r}{\varepsilon}\right) u(x)\, dr = u(x) \int_{\mathbb{R}^n} \rho_\varepsilon(y)\, dy = u(x)$$

Derivative Estimate

Theorem Assume u is harmonic in U.

Then
$$|D^\alpha u(x_0)| \le \frac{C_k}{r^{n+k}} \int_{B(x_0,r)} |u(y)|\,dy = \frac{\tilde{C}_k}{r^k} ⨍_{B(x_0,r)} |u(y)|\,dy$$

for each ball $B(x_0,r) \subseteq U$ and each multi-index α of order $|\alpha| = k$.

proof:
$$u(x) = \int_{B(x,r)} r^{-n} \rho\left(\frac{x-y}{r}\right) u(y)\,dy$$

$$D^\alpha u(x) = \int_{B(x,r)} r^{-n} D_x^\alpha \left[\rho\left(\frac{x-y}{r}\right)\right] u(y)\,dy$$

$$= \int_{B(x,r)} r^{-n}\, r^{-|\alpha|} (D^\alpha \rho)\left(\frac{x-y}{r}\right) u(y)\,dy$$

now $|D^\alpha \rho|$ bounded since $\rho \in C_c^\infty(\mathbb{R}^n)$,
$\le C_{k,d}$

So $$|D^\alpha u(x)| \le C_k r^{-n+k} \int_{B(x,r)} |u(y)|\,dy.$$

Liouville's theorem

note that the ~~est~~ derivative estimates get worse for r small but for the same reason get better for r large

Liouville's Theorem: Suppose $u: \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded. Then u is constant.

proof: Fix $x_0 \in \mathbb{R}^n$

$$|Du(x_0)| \leq \frac{C_1}{r} \fint_{B(x_0,r)} |u(y)|\,dy$$
$$\leq \frac{C_1 \max_{\mathbb{R}^n} |u(y)|}{r}$$

let $r \to \infty \Rightarrow Du(x_0) = 0$.

x_0 arbitrary $\Rightarrow u = \text{constant in } \mathbb{R}^n$.

Thm If u is a bounded ~~harmonic function~~ solution of

$$-\Delta u = f \quad \text{in } \mathbb{R}^n, \quad f \in C_c^2(\mathbb{R}^n)$$

then $$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\,dy + C \quad \text{some } C \in \mathbb{R}.$$

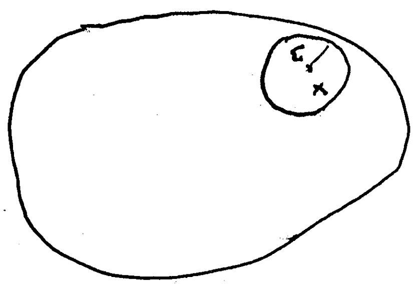
Green's functions

Like for Poisson's equation in $\mathbb{R}^n$ we would like to look for an inversion formula

like $u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$

for boundary value problems.

What happens if we try to use Φ?



U $\quad V_\varepsilon = U \setminus \overline{B(x,\varepsilon)}$

as before

$$\int_{V_\varepsilon} u(y) \Delta \Phi(y-x) - \Phi(y-x) \Delta u(y)\, dy$$

$$= \int_{\partial V_\varepsilon} u(y) \frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x) \frac{\partial u}{\partial \nu}(y)\, dS_y$$

∂V_ε consists of two parts ∂U and $\partial B(x,\varepsilon)$

$$= \int_{\partial B(x,\varepsilon)} u(y) \frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x) \frac{\partial u}{\partial \nu}(y)\, dS_y + \int_{\partial U} u(y) \frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x) \frac{\partial u}{\partial \nu}\, dS_y$$

now, as before, when $\varepsilon \to 0$

$$\int_{\partial B(x,\varepsilon)} u(y) \frac{\partial \Phi}{\partial \nu} dS_y \to u(x) \quad \text{as } \varepsilon \to 0$$

and

$$\int_{\partial B(x,\varepsilon)} \Phi(y-x) \frac{\partial u}{\partial \nu} dS_y \to 0 \quad \text{as } \varepsilon \to 0.$$

so

$$u(x) = -\int_U \Phi(y-x) \Delta u(y) dy + \int_{\partial U} \Phi(y-x) \frac{\partial u}{\partial \nu}(y) - u(y) \frac{\partial \Phi}{\partial \nu}(y-x) dS_y$$

so if we want $\Delta u = f$ then

$$= -\int_U \Phi(x-y) f(y) dy + \underbrace{\int_{\partial U} \Phi(x-y) \frac{\partial u}{\partial \nu}(y) - u(y) \frac{\partial \Phi}{\partial \nu}(x-y) dS_y}$$

this is almost a solution formula but we dont know <u>both</u> $u(y)$ and $\frac{\partial u}{\partial \nu}(y)$ on ∂U

only one of the other (depending on Dirichlet or Neumann problem)

[What we really want in (say) Dirichlet case is a function $G(x,y)$ which has the same property as Φ in that

$$\begin{cases} -\Delta_y G(x,y) = \delta_x & \text{in } y \in U \\ G(x,y) = 0 & y \in \partial U \end{cases}$$

but on ∂U

so that the term

$\int_{\partial U} \Phi(x-y) \frac{\partial u}{\partial \nu}(y)\, dS_y$ does not appear.]

Another way to unfold this, we define a <u>corrector</u> $\phi_x(y)$ solving

$$\begin{cases} -\Delta \phi_x = 0 & \text{in } U \\ \phi_x(y) = \Phi(x-y) & \text{on } \partial U \end{cases}$$

then $\Phi(x-y) - \phi_x(y) = 0$ in U

We <u>define</u> $G(x,y) := \Phi(y-x) - \phi_x(y) \quad (x,y \in U,\ x \neq y)$

then applying same derivation or adding zero to both sides in the form

$$-\int_U \phi^x(y)\,\Delta u(y)\,dy = \int_{\partial U} u(y)\frac{\partial \phi^x}{\partial \nu}(y) - \phi^x(y)\frac{\partial u}{\partial \nu}(y)\,dS_y$$

we obtain

$$u(x) = -\int_U G(x,y)\,\Delta u(y)\,dy - \int_{\partial U} g(y)\frac{\partial G}{\partial \nu_y}(x,y)\,dS_y$$

$$\frac{\partial G}{\partial \nu_y}(x,y) = D_y G(x,y)\cdot \nu(y)$$

Thm Now we can prove the Green's representation formula

$$(D)\quad \begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

Thm If $u \in C^2(\bar{U})$ solves (D) then

$$u(x) = -\int_{\partial U} g(y)\frac{\partial G}{\partial \nu}(x,y)\,dS_y + \int_U G(x,y)\,f(y)\,dy.$$

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formally

$$\begin{cases} -\Delta G = \delta_x & \text{in } U \\ G = 0 & \text{on } \partial U \end{cases}$$

(Symmetry)

Thm For all $x, y \in U$, $x \neq y$

$$G(x,y) = G(y,x)$$

proof: fix $x \neq y$ in U, write

$v(z) := G(x,z)$ $\qquad$ $w(z) := G(y,z)$

solving $\begin{cases} -\Delta v = 0 & \text{in } U\setminus\{x\} \\ v = 0 & \text{on } \partial U \end{cases}$ $\qquad$ $\begin{cases} -\Delta w = 0 & \text{in } U\setminus\{y\} \\ w = 0 & \text{on } \partial U \end{cases}$

let $\varepsilon < \min\{|x-y|, \operatorname{dist}(x,\partial U), \operatorname{dist}(y,\partial U)\}$

$U_\varepsilon := U \setminus \big(B(x,\varepsilon) \cup B(y,\varepsilon)\big)$

apply Green's theorem in U_ε

to get

$$\int_{\partial B(x,\varepsilon)} \overset{(I)}{w\frac{\partial v}{\partial \nu}} - \overset{(II)}{v\frac{\partial w}{\partial \nu}}\, dS_z = \int_{\partial B(y,\varepsilon)} \overset{(III)}{v\frac{\partial w}{\partial \nu}} - \overset{(IV)}{w\frac{\partial v}{\partial \nu}}\, dS_z$$

w is smooth near x so

(II) $\to 0$ as $\varepsilon \to 0$

v smooth near y so

(IV) $\to 0$ as $\varepsilon \to 0$

On the other hand

for term (I)

$$v(z) = G(x,z) = \Phi(x-z) - \phi_x(z)$$

so

$$\int_{\partial B(x,\varepsilon)} w(z) \frac{\partial v}{\partial \nu}(z)\, dS_z = \int_{\partial B(x,\varepsilon)} w(z) \left[\frac{\partial \Phi}{\partial \nu}(x-z) - \frac{\partial \phi_x}{\partial \nu}(z) \right] dS_z$$

now ϕ_x is smooth near z so

$$\int_{\partial B(x,\varepsilon)} w(z) \frac{\partial \phi_x}{\partial \nu}(z)\, dS_z \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

so all that is left is

$$\int_{\partial B(x,\varepsilon)} w(z) \frac{\partial \Phi}{\partial \nu}(x-z)\, dS_z \to w(x) = G(y,x)$$

as $\varepsilon \to 0$

similar argument for RHS yields

$$w(x) = v(y)$$

i.e.

$$G(y,x) = G(x,y).$$

The Method of Images

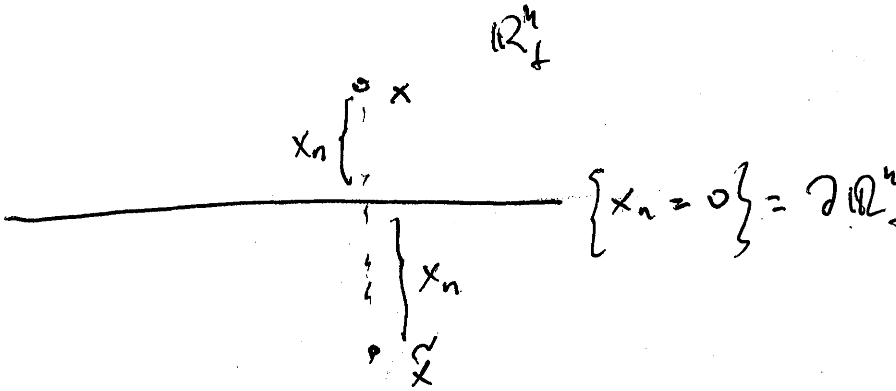
Next we will construct the Green's functions for some simple domains. The upper half space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$ and the unit ball $B(0,1)$.

The idea is to use symmetries to construct the image point $\tilde{x} \in \mathbb{R}^n \setminus \bar{U}$ so that $\Phi(L(x-y))$ exactly ~~cancels~~ is equal to $\Phi(x-y)$ on ∂U.

L is a scaling factor $(\Delta u(\lambda \cdot) = \lambda^2 (\Delta u)(\lambda \cdot))$

Example Half space $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$

define $\tilde{x} = (x_1, \dots, x_{n-1}, -x_n)$ the reflection of x through the plane $\partial \mathbb{R}^n_+$.



~~the claim~~ note that for all $y \in \partial\mathbb{R}^n_+$ $|y-x| = |y-\tilde{x}|$ so

then $\Phi(y-\tilde{x}) = \Phi(y-x)$ on $\partial\mathbb{R}^n_+$

So we set $\phi^x(y) = \Phi(y-\tilde{x})$

$= \Phi(y_1 - x_1, \dots, y_{n-1} - x_{n-1}, y_n + x_n)$

$$G(x,y) = \Phi(y-x) - \underbrace{\Phi(y-\tilde{x})}_{}$$

harmonic in $\mathbb{R}^n \setminus \{\tilde{x}\}$
and hence in $\mathbb{R}^n_+$.

To compute the representation formula need

$\frac{\partial G}{\partial y_n}(x,y)$ on $y_n = 0$

$$\frac{\partial G}{\partial \nu} = -\frac{\partial G}{\partial y_n} = \frac{\partial \Phi}{\partial y_n}(x-y) - \frac{\partial \Phi}{\partial y_n}(\tilde{x}-y)$$

$$-\frac{\partial G}{\partial y_n} = -\frac{1}{n\alpha(n)}\left[\frac{x_n - y_n}{|x-y|^n} - \frac{-x_n - y_n}{|\tilde{x}-y|^n}\right]$$

$$= -\frac{2x_n}{n\alpha(n)}\frac{1}{|x-y|^n} \quad \text{where } y \in \partial\mathbb{R}^n_+$$

Then if u solves

$$\begin{cases} -\Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = g & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

we expect

$$\boxed{u(x) = \frac{2x_n}{n\alpha(n)}\int_{\partial\mathbb{R}^n_+}\frac{g(y)}{|x-y|^n}\,dy \quad (x \in \mathbb{R}^n_+)}$$

$$K(x,y) := \frac{2x_n}{n\alpha(n)}\frac{1}{|x-y|^n} \quad (x \in \mathbb{R}^n_+,\ y \in \partial\mathbb{R}^n_+)$$

is called the <u>Poisson Kernel</u> for the upper half space.

Example 2 $B(0,1)$ the unit ball