Green's function and Distributions

We are interested to find representation formulas for boundary value problems

like

$$(D) \quad \begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

$$(N) \quad \begin{cases} -\Delta u = f & \text{in } U \\ \partial_\nu u = h & \text{on } \partial U \end{cases}$$

analogous to

$$u(x) = (\Phi * f)\Big|_{x} = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$$

for whole space problem.

Let's start by thinking about the 1-d case

$$\begin{cases} -u''(x) = f(x) & \text{in } [0,1] \\ u(0) = u(1) = 0 \end{cases}$$

We can solve this by integrating twice.

~~u(x)=~~

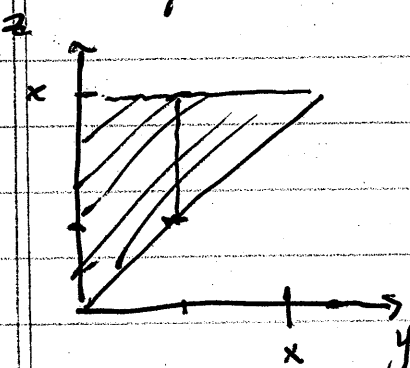
$$-u'(x) = \int_0^x f(y)\,dy + C$$

$$-u(x) = \int_0^x \int_0^z f(y)\,dy\,dz + Cx + D$$

$u(0) = 0$ implies $D = 0$

$u(1) = 0$ implies $C = -\int_0^1 \int_0^z f(y)\,dy\,dz$

It will be more elucidating to change order of integration



$$-u(x) = \int_0^x \int_y^x f(y)\,dz\,dy - \int_0^1 \int_y^1 f(y)\,dz\,dy \; x$$

$$u(x) = \int_0^x (y-x) f(y)\,dy + \int_0^1 x(1-y) f(y)\,dy$$

$$= \int_0^x y(1-x) f(y)\,dy + \int_x^1 x(1-y) f(y)\,dy$$

$$= \int_0^1 G(x,y)\, f(y)\,dy$$

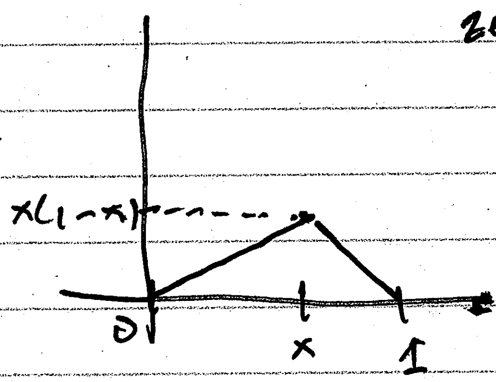
where $G(x,y) = \begin{cases} y(1-x) & 0 \le y \le x \\ x(1-y) & x \le y \le 1 \end{cases}$

fix $x \in (0,1)$ and think of

$G(x,\cdot)$ as a function of y.

graph of $G(x,\cdot)$ has a corner at x of height $x(1-x)$

zero at $0, 1$, linear in between.



Some things to note

$$(Gf)(x) = \int_0^1 G(x,y) f(y)\,dy$$

is an inverse of $Lu = -u''$

$$L : C_0^2([0,1]) \to C([0,1])$$

$$G : C([0,1]) \to C_0^2([0,1])$$

$$(G(Lu))(x) = u(x).$$

$G : [0,1] \times [0,1] \to \mathbb{R}$

1. G is nonnegative
2. G is symmetric $G(x,y) = G(y,x)$
3. G is continuous
4. G is differentiable except on the diagonal $x=y$ where $\frac{\partial G}{\partial y}$ has a jump discontinuity of height 1
5. $-\frac{\partial^2 G}{\partial y^2}(x,y) = 0$ except when ~~$x=y$~~ $y=\cancel{x}$

We will show that 4 and 5 can be interpreted to mean

$$\begin{cases} -\partial_y^2 G(x,y) = \underbrace{\delta(x-y)} \\ G(x,y) = 0 \quad \text{for} \quad y \in \{0,1\} \end{cases}$$

~~It is also~~

What do we mean by $\delta(x)$?

Distribution

The key idea of distribution theory is to stop trying to think of functions on $\mathbb{R}^n$

and instead think of

"~~things~~ objects that can be integrated (or tested) against smooth functions"

Test functions

Call $\mathcal{D} = C_c^\infty(\mathbb{R}^n)$

We make this a topological vector space w/

sequence ϕ_n converges to ϕ in $\mathcal{D}$ as $n \to \infty$ if

1. $\exists$ K cpct so that $\operatorname{supp} \phi_n \subset K$ for all n and $\operatorname{supp} \phi \subset K$

2. $D^j \phi_n \to D^j \phi$ as $n \to \infty$ uniformly on K for every $j \geq 0$.

example

$\rightarrow$ w/ C fixed by $\int_{\mathbb{R}^n} \eta(x)\,dx = 1$

$$\eta(x) = \begin{cases} C e^{-\frac{1}{1-|x|^2}} & \text{if } |x|<1 \\ 0 & |x| \geq 1 \end{cases}$$

this particular test fcn is usually called the standard mollifier.

$$\eta_\varepsilon(x) := \varepsilon^{-n} \eta\left(\frac{x}{\varepsilon}\right)$$

As you saw on HW

Thm Let $f \in C(\mathbb{R}^n)$ and define for $\varepsilon > 0$

$f_\varepsilon(x) = (\eta_\varepsilon * f)(x)$. Then $f_\varepsilon \to f$

pointwise and locally unif in $\mathbb{R}^n$.

Distributions

The space of distributions $\mathcal{D}'$ is

the dual space of $\mathcal{D}$, it is

the space of continuous linear functionals

on $\mathcal{D}$. $f \in \mathcal{D}'$ means

$f : \mathcal{D} \to \mathbb{R}$ and

1. f linear

2. f cts wrt topology on D

we often denote $f(\phi) = (f, \phi)$

<u>examples</u> :

$$(f_1, \phi) = \int_{\mathbb{R}^n} \phi(x)\, dx$$

$$(f_2, \phi) = (\delta, \phi) = f(0)$$

$$(f_3, \phi) = f'(0)$$

~~many dist~~

one class of distributions is given by locally integrable functions

$$\tilde{g} : \mathbb{R}^n \to \mathbb{R}$$

$$(g, \phi) = \int_{\mathbb{R}^n} \tilde{g}(x) \phi(x)\, dx$$

e.g. f_1 above is associated w/

$$\tilde{f}_1(x) = 1$$

~~obv~~

Convergence in $\mathcal{D}'$

(weak -* topology)

$f_n \to f$ if

$(f_n, \phi) \to (f, \phi)$ for every $\phi \in \mathcal{D}$

Derivatives of distribution

how would it work if f

is represented by a C^1 function $\tilde{f}$?

$$(\partial_{x_i} f^{*}, \phi) \underset{\text{should}}{=} \int_{\mathbb{R}^n} \partial_{x_i} \tilde{f}^{*}(x)\, \phi(x)\, dx$$

$$= -\int_{\mathbb{R}^n} \tilde{f}(x)\, \partial_{x_i} \phi^{*}(x)\, dx$$

$$= -(f, \partial_{x_i} \phi^{*})$$

we use this as definition

$$(f', \phi) := -(f, \partial_{x_i} \phi)$$

__Example__ (why the derivative of a function w/ jump discontinuity of height 1 is a δ-function)

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$ __Heaviside function__

on $\mathbb{R}$

$$(H, \phi) = \int_0^\infty \phi(x)\,dx$$

$$(H', \phi) = -(H, \phi')$$

$$= -\int_0^\infty \phi'(x)\,dx$$

$$= -(\phi(\infty) - \phi(0)) \quad (\phi \text{ cpct support})$$

$$= \phi(0)$$

so $H' = \delta$ (as distributions)

of course $(H + \lambda)' = \delta$ for any $\lambda \in \mathbb{R}$ as well.

so now we can agree that in fact

$$-\partial_y^2 G(x,y) = \delta_x(y) \quad \text{(in sense of distributions)}$$

This is why we search for <u>Green's function</u> solving

$$\begin{cases} -\Delta_y G(x,y) = \delta_x & y \text{ in } U \\ G(x,y) = 0 & y \text{ in } \partial U \end{cases}$$

$$u(x) = \int_U G(x,y) f(y) dy$$

$$-\Delta_x u = \int_U -\Delta_x G(x,y) f(y) dy$$

$$\text{(symmetry)} \quad = \int_U -\Delta_y G(x,y) f(y) dy$$

$$= (\delta_x, f)$$

$$= f(x)$$

(formal computation)

~~we can~~:

we have a starting point,

Φ the fundamental solution

solves $-\Delta \Phi = \delta_0$ in $\mathbb{R}^n$

(In fact we proved this by showing

$-\Delta u = f(x)$ for $f \in C_c^\infty(\mathbb{R}^n)$

w/ $u = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$)

only problem is that

$\Phi(y-x)$ is not $=0$ on ∂U.

$\cancel{\Phi(y_0 - x) =}$

to correct this we define

corrector $\phi_x(y)$ solving

$$\begin{cases} -\Delta \phi_x = 0 & \text{in } U \\ \phi_x(y) = \Phi(y-x) & \text{on } \partial U \end{cases}$$

then define

$$\boxed{G(x,y) = \Phi(y-x) - \phi_x(y)}$$