Laplace Equation / Poisson Equation (Chapter 8)

Office Hours
M ~~4-5 pm~~ 2:30-3:30 pm
T 4-5 pm
this week
ThF 1:30-2:30 pm

$-\Delta u = 0$ Laplace's Equation

$-\Delta u = f(x)$ Poisson's Equation.

e.g. in electrostatics or Newtonian gravitation
u is electrostatic potential
or Gravitational potential
Du is electric field
gravitational field
f is charge ~~distribution~~ density
or mass ~~distribution~~ density

The fundamental solution

As a first step in understanding an equation
it is often useful to ~~look for identify~~
write down at least one explicit solution.
~~Usually it is a~~

We will look for radially symmetric solutions

$$u(x) = v(r) \quad \text{w/} \quad r = (x_1^2 + \cdots + x_n^2)^{1/2}$$

~~In radial polar coordinates the laplacian can be written~~

Δ

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} \qquad x \neq 0$$

so
$$u_{x_i} = v'(r)\frac{\partial r}{\partial x_i} = v'(r)\frac{x_i}{r}$$

$$u_{x_i x_i} = v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i}{r^2}\frac{x_i}{r}\right)$$

and
$$\sum_{i=1}^{n} u_{x_i x_i} = \frac{1}{r^2}v''(r)\underbrace{\left(\sum_i x_i^2\right)}_{= r^2} + v'(r)\left(\frac{n}{r} - \frac{\sum_i x_i^2}{r^3}\right)$$

$$= v''(r) + v'(r)\frac{n-1}{r}$$

so $-\Delta u = 0$ means for v,

$$v'' + \frac{n-1}{r}v' = 0$$

or
$$\frac{v''}{v'} = -\frac{n-1}{r}$$

using $(\log v')' = \frac{v''}{v'}$

$$\log v' = -(n-1)\log r + c$$

or exponentiating both sides

$$v' = \frac{A}{r^{n-1}} \quad \text{(with new } A = \log C$$

then integrating again

$$v(r) = \begin{cases} A \log r + B & \text{if } n = 2 \\ \frac{A}{r^{n-2}} + B & \text{if } n \geq 3. \end{cases}$$

We define the <u>fundamental solution</u> Φ choosing a particular pair of constants which will be natural later

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & \text{if } n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & \text{if } n \geq 3 \end{cases}$$

$\alpha(n)$ = volume of unit ball in $\mathbb{R}^n$

from construction $-\Delta \Phi = 0$ in $\mathbb{R}^n \setminus \{0\}$

[it will turn out that, based on our choice of normalization,

$- \Delta \Phi = \delta_0$ ← delta mass at origin] notice that, since the PDE is translation invariant,

$\Phi(x-y)$ also solves

$$-\Delta \Phi(\cdot - y) = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \{y\}$$

if we have $f: \mathbb{R}^n \to \mathbb{R}$

then $\Phi(x-y) f(y)$ is also harmonic in $\mathbb{R}^n \setminus \{y\}$

so would be $\sum_{j=1}^{m} \Phi(x-y_j) f(y_j)$

for any finite collection $(y_j)_{j=1}^{m} \subseteq \mathbb{R}^n$.

harmonic in $\mathbb{R}^n \setminus \{y_1, \dots, y_m\}$.

for $f: \mathbb{R}^n \to \mathbb{R}$, C^2 w/ compact support

($\cancel{\{f \neq 0\}}$ ~~is~~ $\overline{\{f \neq 0\}}$ is compact)

define $$\boxed{u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy = (\Phi * f)(x)}$$

this is called the convolution of Φ and f

note that $|D\Phi(x)| \le \frac{C}{|x|^{n-1}}$, $\underbrace{|D^2\Phi(x)| \le \frac{C}{|x|^n}}$

non-integrable singularity at origin.

so we are not allowed to just differentiate under the integral sign so easily.

<u>Thm</u> If $f \in C_c^2(\mathbb{R}^n)$ and $u(x)$ given by

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$$

then u solves Poisson's Equation

$$-\Delta u(x) = f(x) \quad \text{in } \mathbb{R}^n .$$

<u>proof</u>: first we make a change of variables in the integral

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy = \int_{\mathbb{R}^n} f(x-z)\, \Phi(z)\, dz$$

call $z = x-y$ (then $y = x-z$) $= (f * \Phi)(x)$

(Note that we just showed that convolution is commutative operation)

now since f is C_c^2 we are allowed to differentiate under the integral to get

$$+\Delta u(x) = \int_{\mathbb{R}^n} +\Delta_x(f(x-y))\,\Phi(y)\,dy$$

$$= \int_{\mathbb{R}^n} +\Delta_y(f(x-y))\,\Phi(y)\,dy$$

now the idea is to integrate by parts to put the Laplacian back on Φ (at least away from the singularity)

for $\varepsilon > 0$ call $U_\varepsilon = \mathbb{R}^n \setminus \overline{B(0,\varepsilon)}$

($B(x,r)$ is ball of radius r centered at x)

$$+\Delta u(x) = \underbrace{\int_{B(0,\varepsilon)} \Phi(y)\,\Delta_x f(x-y)\,dy}_{\text{I)}} + \underbrace{\int_{U_\varepsilon} \Phi(y)\,\Delta_y f(x-y)\,dy}_{\text{II}}$$

We claim that (I) $\to 0$ as $\varepsilon \to 0$,

$|\Delta_x f(x-y)| \leq \sup_{\mathbb{R}^n} |\Delta f| < \infty$ given $f \in C^2_c(\mathbb{R}^n)$

and $\Phi(y)$ has integrable singularity

$$\left| \int_{B(0,\varepsilon)} \Phi(y)\, \Delta_x f(x-y)\, dy \right| \leq C \int_{B(0,\varepsilon)} |\Phi(y)|\, dy$$

$$= C\, n\alpha(n) \int_0^{\varepsilon} \frac{1}{n(n-2)\alpha(n)} \frac{1}{r^{n-2}} r^{n-1}\, dr$$

$$= \frac{C}{n-2} \cancel{n\alpha(n)} \int_0^{\varepsilon} r\, dr = C\varepsilon^2$$

for the second term (II) we need to integrate by parts. Since this is the first of many times we will do this I will write the identity.

Aside

Divergence theorem

domain U, outer normal ν $\vec{f}$ vector field

$$\int_U \operatorname{div} \vec{f}\, dx = \int_{\partial U} \vec{f} \cdot \nu\, dS$$

if $\vec{f}$ vector field, g scalar fn on U

$$\int_U \operatorname{div}(\vec{f} g)\, dx = \int_U g \operatorname{div} \vec{f} + \vec{f} \cdot \nabla g$$

div thm yields the identity

$$\int_U g \operatorname{div} \vec{f} + \vec{f} \cdot \nabla g = \int_{\partial U} g\, \vec{f} \cdot \nu\, dS .$$

applying this for some functions $u, v : U \to \mathbb{R}$, C^2

$(\vec{F} = \nabla v)$

$$\int_U u \, \Delta v = \int_{\partial U} u \frac{\partial v}{\partial \nu} \, dS - \int_U \nabla u \cdot \nabla v$$

also

$$\int_U v \, \Delta u = \int_{\partial U} v \frac{\partial u}{\partial \nu} \, dS - \int_U \nabla u \cdot \nabla v$$

~~this~~ subtracting yields

Green's Identity

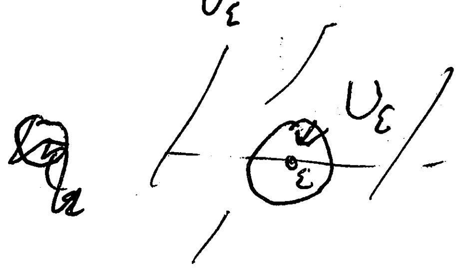
$$\boxed{\int_U u \Delta v - v \Delta u \, dx = \int_{\partial U} u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \, dS}$$

---

back to the proof we are considering

$$\Delta u(x) = O(\varepsilon^2) + \underbrace{\int_{U_\varepsilon} \Delta_y f(x-y) \, \Phi(y) \, dy}_{(II)}$$

$$(II) = \int_{U_\varepsilon} f(x-y) \, \Delta \Phi(y) \, dy + \int_{\partial B(0,\varepsilon)} f(x-y) \frac{\partial \Phi}{\partial \nu_y} - \Phi(y) \frac{\partial f}{\partial \nu_y}(x-y)$$



outward unit normal points in to $B(0,\varepsilon)$ to U_ε

now $\Delta\Phi = 0$ in $\mathbb{R}^n \setminus B(0,\varepsilon)$ so

$$(\mathrm{II}) = \int_{\partial B(0,\varepsilon)} \underbrace{\Phi(y)\frac{\partial f}{\partial \nu_y}(x-y)}_{(\mathrm{III})} - \underbrace{f(x-y)\frac{\partial \Phi}{\partial \nu_y}(y)}_{(\mathrm{IV})}\, dS_y$$

now $\nu_y = \frac{-y}{|y|} = \frac{-y}{\varepsilon}$ on is outer unit normal

to U_ε on $\partial B(0,\varepsilon)$

i.e. $$\frac{\partial \Phi}{\partial \nu_y} = -\frac{\partial \Phi}{\partial r}(y) = +\frac{1}{n\alpha(n)}\frac{1}{|y|^{n-1}} = \frac{+1}{n\alpha(n)}\frac{1}{\varepsilon^{n-1}}$$

~~while~~ $\cancel{|\Phi(y)|}$

so $$(\mathrm{IV}) = \frac{-1}{\varepsilon^{n-1} n\alpha(n)}\int_{\partial B(0,\varepsilon)} f(x-y)\, dS_y \cancel{= f(x) + \frac{1}{\varepsilon^{n-1}n\alpha(n)}\int_{\partial B(0,\varepsilon)} f(x-y)-f(x)}$$

since f is continuous this term $\to 0$ as $\varepsilon \to 0$.

$$= -f(x) + \frac{1}{n\alpha(n)\varepsilon^{n-1}}\overbrace{\int_{\partial B(0,\varepsilon)} -f(x-y) + f(x)\, dS_y}$$

so $(\mathrm{IV}) \to -f(x)$ as $\varepsilon \to 0$

$(\mathrm{III}) \to 0$ as $\varepsilon \to 0$ since $\frac{\partial f}{\partial \nu_y}$ is bdd

and Φ is integrable.

for $n \geq 3$ we compute to check, $\Phi(y) = \frac{1}{n\alpha(n)(n-2)}\frac{1}{\varepsilon^{n-2}}$

$$\left|\int_{\partial B(0,\varepsilon)} \Phi(y)\frac{\partial f}{\partial \nu_y}(x-y)\, dS_y\right| \leq \left(\sup |Df|\right)\frac{n\alpha(n)\varepsilon^{n-1}}{n(n-2)\alpha(n)}\varepsilon^{2-n} = O(\varepsilon)$$

thus we obtain the desired result

~~to~~ $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^n$. $\square$

We solved Poisson's equation in $\mathbb{R}^n$

when $f \in C^2_c(\mathbb{R}^n)$! (Existence)

<u>Interpretation:</u> $-\Delta\Phi = \delta_0$, $-\Delta\Phi(x-y) = \delta_x(y)$

$-\Delta u(x) = \int_{\mathbb{R}^n} -\Delta\Phi(x-y) f(y)\,dy = \int_{\mathbb{R}^n} \delta_x(y) f(y)\,dy = f(x)$

<u>Properties of Harmonic functions</u>

let u be a solution of

$-\Delta u = 0$ in U called <u>harmonic</u> in U

U - open ~~bounded~~ subset of $\mathbb{R}^n$.

~~Theorem (Maximum Principle~~

<u>Thm</u> (Mean Value Property) ~~Suppose u is harmonic~~

~~in $U \subset \mathbb{R}^n$ open set~~, Suppose $u \in C^2(U)$ then

u is harmonic in U if and only if

it has the MVP, for all $x \in U$ and

$r > 0$ so that $B(x,r) \subseteq U$,

$$\boxed{u(x) = ⨍_{\partial B(x,r)} u(y)\,dS_y = ⨍_{B(x,r)} u(y)\,dy}$$

proof: Suppose u is harmonic in U, $B(x,r) \subseteq U$,

$$0 = \int_{B(x,r)} \Delta u(y)\, dy = \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}\, dS$$

$$= \int_{\partial B(x,r)} \nabla u(y) \cdot \frac{y-x}{r}\, dS_y$$

Call $\phi(r) = ⨍_{\partial B(x,r)} u(y)\, dS_y$

$\lim_{r \to 0} \phi(r) = u(x)$, we claim that ϕ is constant for $r < \text{dist}(x, \partial U)$

change variables to $y = x + rz$

$$\phi(r) = ⨍_{\partial B(0,1)} u(x+rz)\, dS_z$$

(note that the r^{n-1} for Jacobian of the change of variables cancels w/ the $\frac{1}{r^{n-1}}$ in $⨍_{\partial B(x,r)}$)

$$\phi'(r) = ⨍_{\partial B(0,1)} z \cdot \nabla u(x+rz)\, dS_z$$

$$= ⨍_{\partial B(x,r)} \frac{y-x}{r} \cdot \nabla u(\cancel{x+rz}\; y)\, dS_y = \frac{\int_{B(x,r)} \Delta u(y)\, dy}{|\partial B(x,r)|} = 0$$

So, since $\phi'(r) = 0$,

$$\Rightarrow \quad \phi(r) = \phi(0) = u(x).$$

We just integrate this to obtain the $⨍_{B(x,r)}$ case ~~Similar argum~~

$$\int_{B(x,r)} u(y)\,dy = \int_0^r \int_{\partial B(x,\rho)} u(y)\,dS\,d\rho$$

$$= u(x) \int_0^r n\alpha(n)\rho^{n-1}\,d\rho$$

$$= u(x)\, n\alpha(n) \left(\frac{\rho^n}{n}\right)_0^r$$

$$= \alpha(n) r^n u(x) = |B(x,r)|\, u(x) \qquad \blacksquare$$

Now we want to show that

MVP $\Rightarrow$ harmonic.

Same calculation shows

$$0 = \phi'(r) = \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} \Delta u(y)\,dy$$

$\subset_{MVP}$

$$= \frac{r}{n} ⨍_{B(x,r)} \Delta u(y)\,dy$$

thus $⨍_{B(x,r)} \Delta u(y)\,dy = 0$ for every $x \in U$, $r > 0$ suff small

taking $r \to 0 \Rightarrow \Delta u(x) = 0$ since Δu is continuous $\square$

So actually the mean value property is equivalent to being harmonic.

This is a special property of Laplace operator that does not generalize to other elliptic equations.

The next property however ~~is not~~ is a general feature of elliptic equations.

Thm (Maximum principle) Let $U \subseteq \mathbb{R}^n$ open and bounded. Let $u \in C^2(U) \cap C(\overline{U})$ harmonic. Then

1. weak max: $\max_{x \in \overline{U}} u(x) = \max_{x \in \partial U} u(x)$

2. Strong max: If U connected then either $u =$ constant in $\overline{U}$ <u>or</u>

$u(x) < \max_{y \in \partial U} u(y)$ for all $x \in U$.

Notes: • same for ~~minimums~~ minimum since $-u$ harmonic as well proof 1: we just prove the strong form, weak follows

Suppose there is $x_0 \in U$ s.t.

$$u(x_0) = \max_{x \in \overline{U}} u(x) = M .$$

let $r > 0$ so that $B(x_0, r) \subseteq U$

MVP $\Rightarrow$ $\quad u(x_0) = M = ⨍_{B(x_0,r)} u(y)\, dy$

but $u(y) \leq M$ everywhere

if u were strictly $< M$ some point in $B(x_0,r)$ then $⨍_{B(x_0,r)} u(y)\, dy < M$

$$\Rightarrow \quad u(x) = M \quad \text{in } B(x_0, r)$$

Thus we have shown

$S = \{x \in \overline{U} : u(x) = M\}$ is <u>open</u>

it is also closed since u is cts in $\overline{U}$

thus, since $\overline{U}$ is connected,

$$S = \overline{U} .$$

proof 2: (this is the proof that generalizes better, also it uses the PDE in a more direct way.)

Suppose that there is $x_0 \in U$ so that

$$u(x_0) = \max_{x \in \bar{U}} u(x) = M$$

then $Du(x_0) = 0$ and $D^2u(x_0) \leq 0$ (in sense of matrices)

then all $u_{x_i x_i}(x_0) \leq 0$

and $\Delta u(x_0) \leq 0$

this isn't quite a contradiction of the PDE,

We need to perturb a little to make the inequality strict and get a contradiction.

Suppose $u(x_0) = \max_{x \in \bar{U}} u(x) = M > \max_{x \in \partial U} u(x)$

consider $u(x) + \frac{\delta}{2n}|x|^2 = v(x)$

if δ is sufficiently small,

$$\frac{\delta}{2n} \operatorname{diam}(U)^2 < M - \max_{x \in \partial U} u(x)$$

then v will still have an interior maximum in U

since ~~$v(x_0) \geq u(x_0)$~~

$$v(x_0) \geq u(x_0) = \max_{\partial U} u + \left(M - \max_{\partial U} u\right)$$

$$\geq \max_{\partial U} v - \frac{\delta}{2n}(\mathrm{diam}\, U)^2 + (M - \max u)$$

$$> \max_{\partial U} v .$$

so there is a new point $\tilde{x}_0$ so that v has a local max at $\tilde{x}_0 \in U$

$$\Rightarrow \quad Dv(\tilde{x}_0) = 0 \qquad \Delta v(\tilde{x}_0) \leq 0$$

<u>but</u>

$$\Delta v(\tilde{x}_0) = \Delta u(\tilde{x}_0) + \Delta\left(\frac{\delta}{2n}|x|^2\right)\Big|_{x=\tilde{x}_0}$$

$$= 0 + \delta > 0 .$$

this is a contradiction.

$\square$