Hamilton - Jacobi Equations

$$(1) \quad \begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) & x \in \mathbb{R}^n \end{cases}$$

$H : \mathbb{R}^n \to \mathbb{R}$ called the Hamiltonian

our goal as w/ the conservation laws is to continue the solution past the time when characteristics cross. To do so it will be necessary, as before, to select the correct weak solution ~~correspo~~ by using some physical principle.

We will find that (1) PDE is associated w/ an optimal control problem

Characteristics for (1) are

$$\begin{cases} \dot{x} = D_p H(p,x) \\ \dot{p} = -D_x H(p,x) \end{cases} \quad \text{called Hamilton's ODEs}$$

these ODE arise in classical mechanics from a variational principle

Let $L : \mathbb{R}^n \to \mathbb{R}$ a given smooth fn called a Lagrangian. We write $L(\check{q}, x)$,

or Running Cost

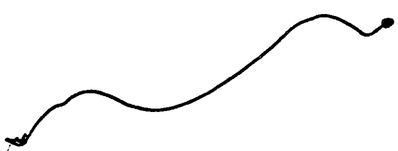
fix $x, y \in \mathbb{R}^n$ and $t > 0$ the action

or cost functional is defined

$$I[w] = \int_0^t L(\dot{w}(s), w(s))\, ds$$ defined for

$$w \in \mathcal{A} = \{ w(\cdot) \in C^2([0,t]; \mathbb{R}^n) : w(0) = y, w(t) = x \}$$

smooth paths from y to x



x

y

we wish to find the minimal cost/action path from y to x.

$$I[x(\cdot)] = \min_{w \in \mathcal{A}} I[w(\cdot)] .$$

Let us assume we can find such a path,

$x(\cdot)$

then ~~Ф~~

Thm (Euler-Lagrange Eqns) The fcn x solves

$$\boxed{-\frac{d}{ds}\left(D_q L(\dot{x}, x)\right) + D_x L(\dot{x}, x) = 0 \qquad 0 \le s \le t.}$$

proof: as we did for previous variational principles we look at the "directional derivative" of I in direction $\phi \in \tan_x A$.

let ϕ smooth $[0,t] \to \mathbb{R}^n$,

$\phi(0) = \phi(t) = 0$ so that

$w = x + \varepsilon\phi \in A$

~~$I[w]$~~ $I[x] \le I[w] \quad \forall \varepsilon > 0.$

so $i(\varepsilon) = I[x + \varepsilon\phi]$ has a min at $\varepsilon = 0$

$\Rightarrow \quad i'(0) = 0$ if it exists.

$i(\varepsilon) = \int_0^t L(\dot{x} + \varepsilon\dot{\phi}, x + \varepsilon\phi) \, ds$ so

$i'(\varepsilon) = \int_0^t D_q L(\dot{x} + \varepsilon\dot{\phi}, x + \varepsilon\phi)\dot{\phi} + D_x L(\dot{x} + \varepsilon\dot{\phi}, x + \varepsilon\phi)\phi \, ds$

setting $\varepsilon = 0$

$$0 = i'(0) = \int_0^t D_q L(\dot{x}, x)\dot{\phi} + D_x L(\dot{x}, x)\phi \, ds$$

and integrate by parts using

$$\phi(0) = \phi(t) = 0$$

$$= \int_0^t \left(-\frac{d}{ds}(D_q L(\dot{x}, x)) + D_x L(\dot{x}, x) \right) \phi \, ds$$

Since this is zero for all $\phi \in C_c^2((0,t) : \mathbb{R}^n)$

we obtain

$$\begin{cases} -\frac{d}{ds}(D_q L(\dot{x}, x)) + D_x L(\dot{x}, x) = 0 \\ \text{for } s \in (0, t) \end{cases}$$

$\square$

Example 1: $L(q, x) = \frac{1}{2} m |q|^2 - \phi(x)$

kinetic energy   potential energy

E-L eqn is Newton's law

$$\boxed{m\ddot{x} = -D\phi(x)}$$

Example 2: $L(q, x) = \begin{cases} 0 & |q| \le 1 \\ +\infty & |q| > 1 \end{cases}$

cost functional is minimal time to move from y to x w/ speed ≤ 1. E-L eqn is harder to interpret.

A payoff problem

define

$$u(x,t) = \inf\left\{ \int_0^t L(\dot{w}(s))\,ds + g(y) \;\middle|\; w(0)=y,\ w(t)=x \right\}$$

the running cost

the value function

the payoff / terminal cost

optimal trajectories solve the Euler-Lagrange eqn

$$\frac{d}{ds}\left(D_v L(\dot{x})\right) = 0 \qquad 0 \le s < t.$$

[dynamic ~~programming~~ principle], $\forall\ 0 < s \le t$,

$$u(x,t) = \inf\left\{ \int_s^t L(\dot{w}(\tau), w(\tau))\,d\tau + u(y,s) \;\middle|\; w(s)=y,\ w(t)=x \right\}$$

using the DPP infinitesimally

$$u(x,t+h) = \inf_w \left\{ \int_t^{t+h} L(\dot{w}(\tau), w(\tau))\,d\tau + u(y,t) \;\middle|\; w(t)=y,\ w(t+h)=x \right\}$$

$$= \inf_v \left\{ hL(v,x) + u(x-hv,t) \right\} + O(h^2)$$

$$= \inf_v \left\{ hL(v,x) + u(x,t) - h\,v\cdot Du(x,t) \right\} + O(h^2)$$

$$= u(x,t) + h \inf_v \left\{ -v\cdot Du(x,t) + L(v,x) \right\} + O(h^2)$$

on

$$\partial_t u = \inf_v \{ -v \cdot Du(x,t) + L(v,x) \}$$

or

$$\partial_t u + \underbrace{\sup_v \{ v \cdot Du(x,t) - L(v,x) \}}_{=: H(Du,x)} = 0$$

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proof of DPP :

$$u(x,t) = \inf \left\{ \int_0^t L(\dot{w}, w)\, d\tau + g(y) \mid w(0)=y,\ w(t)=x \right\}$$

(9) let $y \in \mathbb{R}^n$, take w_* optimal for $u(y,s)$, i.e.

$$u(y,s) = \int_0^s L(\dot{w}_*, w_*)\, d\tau + g(w_*(0))$$

and let $\bar{w}(\tau) = \begin{cases} w(\tau) & s \le \tau \le t \\ w_*(\tau) & 0 \le \tau \le s \end{cases}$

with w any path s.t. $w(s)=y$, $w(t)=x$

then $$u(x,t) \le \int_0^t L(\dot{\bar{w}}, \bar{w})\, d\tau + g(\bar{w}(0))$$

$$= u(y,s) + \int_s^t L(\dot{w}(\tau), w(\tau))\, d\tau$$

taking inf over w yields

$$u(x,t) \le \inf \left\{ \int_s^t L(\dot{w}(\tau), w(\tau))\, d\tau + u(y,s) \mid w(s)=y,\ w(t)=x \right\}$$

for the other direction let w_* optimal

for
$$u(x,t) = \int_0^t L(\dot{w}_*, w_*)\,d\tau + g(w_*(0))$$
$$= \int_s^t L(\dot{w}_*, w_*)\,d\tau + \int_0^s L(\dot{w}_*, w_*)\,d\tau + g(w_*(0))$$
$$\geq \int_s^t L(\dot{w}_*, w_*)\,d\tau + u(w_*(s), s)$$

$$\Rightarrow \quad u(x,t) \geq \inf \left\{ \int_s^t L(\dot{w}, w)\,d\tau + u(y,s) : w(s) = y,\ w(t) = x \right\}$$

$\square$

Hamilton's ODE's

Suppose that x is an action minimizer (or critical point) call

$$p(s) = D_v L(\dot{x}(s), x(s)) \quad 0 \le s \le t \,,$$

$p(\cdot)$ is called the generalized momentum

x - position, $\dot{x}$ - velocity.

Now we suppose that $\forall\ x, p \in \mathbb{R}^n$

$$\left\{ \begin{array}{l} p = D_v L(v, x) \text{ can be uniquely solved} \\ \text{for } v \text{ as a smooth fcn of } p, x \\ \qquad v = v(p, x) \,. \end{array} \right.$$

<u>Def</u> The <u>Hamiltonian</u> associated with the Lagrangian L is

$$H(p, x) = p \cdot v(p, x) - L(v(p, x), x)$$

Legendre Transform

Now we put some additional assumptions

(1) $\overset{\vee}{q} \mapsto L(\overset{\vee}{q})$ is convex

(2) $\lim_{|q| \to \infty} \frac{L(v)}{|v|} = +\infty$ <u>coercivity</u>

Def The <u>Legendre transform</u> of L is

$$L^*(p) = \sup_{q \in \mathbb{R}^n} \left\{ p \cdot q - L(q) \right\} \qquad (p \in \mathbb{R}^n)$$

---

suppose the supremum is achieved by $q_* \in \mathbb{R}^n$

$L^*(p) = p \cdot q_* - L(q_*)$ and

$q \mapsto p \cdot q - L(q)$ has maximum at q_*

then $p = DL(q)$ is solvable for q

(q_* solves) although not necessarily

uniquely $L^*(p) = \cancel{p \cdot q(p) - L(q(p))}$

$p \cdot q(p) - L(q(p))$

which is the Hamiltonian H associated w/ L.

we thus call $H = L^*$.

Now conversely given Hamiltonian H

can we find L?

<u>Thm</u> (Convex duality) Assume L convex, coercive

and define $H = L^*$ then

(1) H is convex and coercive

(2) $H^* = L$

Rem: We call H and L convex dual functions

<u>proof</u>:

- $p \mapsto p \cdot q - L(q)$ is linear

so $H(p) = \sup_{q \in \mathbb{R}^n} \{ p \cdot q - L(q) \}$ is convex.

- $H(p) = \sup_{q \in \mathbb{R}^n} \{ q \cdot p - L(q) \}$

take $q = R\frac{p}{|p|}$ $\geq R|p| - L(R\frac{p}{|p|})$

$\geq R|p| - \max_{B(0,R)} L$ (coercive)

$\Rightarrow \lim_{|p| \to \infty} \frac{H(p)}{|p|} \geq R$ for every $R > 0$

(duality):

$\forall$ p, q

$$H(p) + L(q) \geq p\cdot q$$

so $$L(q) \geq \sup_{p\in\mathbb{R}^n}\{p\cdot q - H(p)\} = H^*(q)$$

and

$$H^*(q) = \sup_{p\in\mathbb{R}^n}\Big\{p\cdot q - \sup_{r\in\mathbb{R}^n}\{r\cdot p - L(r)\}\Big\}$$

$$= \sup_{p\in\mathbb{R}^n}\inf_{r\in\mathbb{R}^n}\{p\cdot(q-r) + L(r)\}$$

Since L convex $\Rightarrow$ it has a supporting hyperplane at q, $\exists$ $s\in\mathbb{R}^n$ s.t.

$$L(r) \geq L(q) + s\cdot(r-q) \quad \forall\ r\in\mathbb{R}^n$$

taking $p=s$ in the supremum

$$H^*(q) \geq \inf_{r\in\mathbb{R}^n}\{s\cdot(q-r) + L(r)\}$$

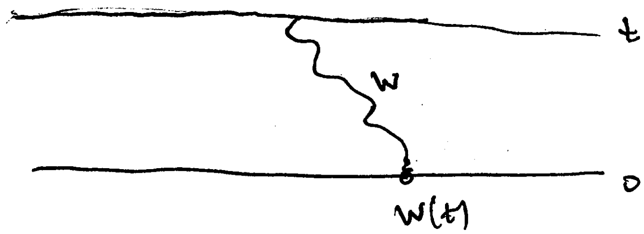
$$\geq L(q) \qquad \text{from choice of } s.$$

$\blacksquare$

The Hopf-Lax formula

define

(1) $u(x,t) = \inf \left\{ \int_0^t L(\dot{w}(s))\,ds + g(y) \;\middle|\; w(0)=y,\ w(t)=x \right\}$



will turn out to solve (in a sense)

$$\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) \end{cases}$$

where $H = L^*$ or $L = H^*$ depending on our starting point.

Thm (Hopf-Lax formula)

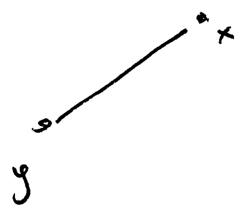
Let $x \in \mathbb{R}^n$, $t > 0$ then u as in (1) is

$$\boxed{u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ t L\left(\frac{x-y}{t}\right) + g(y) \right\}}$$

(i.e. optimal trajectories in (1) are straight lines) proof of Hopf-Lax formula:

one direction is easy

take $w(s) = y + \frac{s}{t}(x-y) \qquad 0 \le s \le t$



$$u(x,t) \le \int_0^t L(\dot{w}(s))\,ds + g(y)$$
$$= \int_0^t L\left(\frac{x-y}{t}\right) ds + g(y)$$
$$= t\,L\left(\frac{x-y}{t}\right) + g(y)$$

take inf over $y \in \mathbb{R}^n$ to obtain

$$u(x,t) \le \inf_{y \in \mathbb{R}^n} \left\{ t\,L\left(\frac{x-y}{t}\right) + g(y) \right\}$$

if w is any C^1 fcn $w(t) = x$ then

calling $y = w(0)$

$$L\left(\frac{1}{t}\int_0^t L(\dot{w}(s))\,ds\right) \le \frac{1}{t}\int_0^t L(\dot{w}(s))\,ds \qquad \text{(Jensen)}$$

so

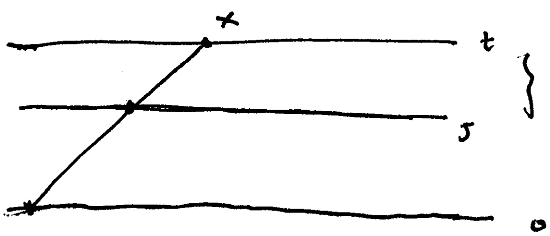
$$u(x,t) \ge \inf_{y \in \mathbb{R}^n} \left\{ t\,L\left(\frac{1}{t}\int_0^t \dot{w}(s)\,ds\right) + g(y) \right\}$$
$$= \inf_{y \in \mathbb{R}^n} \left\{ t\,L\left(\frac{x-y}{t}\right) + g(y) \right\} \qquad \square$$

(use) super-linear growth of L to show inf is achieved.

Lem (Dynamic Programming Principle)

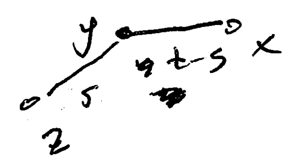
For each $x \in \mathbb{R}^n$, $0 \le s < t$ we have

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ (t-s) L\left(\frac{x-y}{t-s}\right) + u(y,s) \right\}$$



proof: 1. Fix $y \in \mathbb{R}^n$, $0 < s < t$ and choose z so that

$$u(y,s) = sL\left(\frac{y-z}{s}\right) + g(z)$$



Since L convex, $\frac{x-z}{t} = \left(1-\frac{s}{t}\right)\frac{x-y}{t-s} + \frac{s}{t}\frac{y-z}{s}$

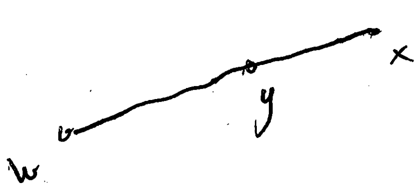
$$L\left(\frac{x-z}{t}\right) \le \left(1-\frac{s}{t}\right) L\left(\frac{x-y}{t-s}\right) + \frac{s}{t} L\left(\frac{y-z}{s}\right) \quad \text{so}$$

$$u(x,t) \le tL\left(\frac{x-z}{t}\right) + g(z) \le (t-s)L\left(\frac{x-y}{t-s}\right) + sL\left(\frac{y-z}{s}\right) + g(z)$$

$$= (t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s)$$

for each $y \in \mathbb{R}^n$.

Now choose w s.t. $u(x,t) = tL\left(\frac{x-w}{t}\right) + g(w)$



set $y = \frac{s}{t}x + \left(1-\frac{s}{t}\right)w$

Lemma (Lipschitz estimate) The function u is Lipschitz continuous in $\mathbb{R}^n \times (0,\infty)$ and $u = g$ on $\mathbb{R}^n \times \{t=0\}$

proof: Fix $t>0$, $x, \hat{x} \in \mathbb{R}^n$. Find $y \in \mathbb{R}^n$ s.t.

$$t L\left(\frac{x-y}{t}\right) + g(y) = u(x,t)$$

then

$$u(\hat{x},t) - u(x,t) = \inf_z \left\{ tL\left(\frac{\hat{x}-z}{t}\right) + g(z) \right\} - tL\left(\frac{x-y}{t}\right) - g(y)$$

$$\left(\text{take } z = \hat{x}-x+y\right) \leq g(\hat{x}-x+y) - g(y) \leq \mathrm{Lip}(g)\, |\hat{x}-x|$$

interchanging roles of $\hat{x}, x$ $\Rightarrow$ $|u(\hat{x},t) - u(x,t)| \leq \mathrm{Lip}(g)\, |\hat{x}-x|$

Now for time continuity, we just prove at $t=0$.

Let $t>0$. Choosing $y=x$ in Hopf-Lax

$$u(x,t) \leq tL(0) + g(x) \quad \text{and}$$

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}$$

$$\geq g(x) + \min_{y \in \mathbb{R}^n} \left\{ -\mathrm{Lip}(g)|x-y| + tL\left(\frac{x-y}{t}\right) \right\}$$

$$\left(z = \frac{x-y}{t}\right) \quad = g(x) - t \max_{z \in \mathbb{R}^n} \left\{ \mathrm{Lip}(g)|z| - L(z) \right\}$$

$$= g(x) - t \max_{w \in B(0,\mathrm{Lip}(g))} \max_{z \in \mathbb{R}^n} \left\{ w \cdot z - L(z) \right\}$$

$$= g(x) - t \max_{w \in B(0, \mathrm{Lip}(g))} H(w)$$

so $|u(x,t) - g(x)| \leq Ct$ w/

$$C = \max\left(|L(0)|, \max_{B(0, \mathrm{Lip}(g))} |H|\right).$$

Using Dynamic Programming principle and Lipschitz continuity in space of $u(x,t)$ $t>0$ yields time Lipschitz continuity. $\square$

Now by Rademacher's thm Lipschitz cts fns are differentiable a.e.

Thm Let $x \in \mathbb{R}^n$, $t>0$ and u defined by Hopf-Lax formula differentiable at (x,t). Then

$$u_t(x,t) + H(Du(x,t)) = 0$$

Proof: ~~$u(x,t)$~~ Fix $q \in \mathbb{R}^n$, $h>0$, by DPP

$$\cancel{u(x+hq, t+h) = \min_{y \in \mathbb{R}^n} \left\{ hL\left(\frac{x+hq-y}{h}\right) + u(y,t) \right\}}$$

$\longrightarrow$

subsolution:

$$u(x+hq,t+h) = \min_{y\in\mathbb{R}^n}\left\{ hL\left(\frac{x+hq-y}{h}\right) + u(y,t)\right\}$$

$(y=x)$ $\qquad \leq hL(q) + u(x,t)$

So $$\frac{u(x+hq,t+h) - u(x,t)}{h} \leq L(q)$$

since u differentiable at x,t

$$q\cdot Du(x,t) + u_t(x,t) \leq L(q)$$

valid for all q so

$$u_t(x,t) + \underbrace{\max_{q\in\mathbb{R}^n}\left\{q\cdot Du(x,t) - L(q)\right\}}_{\displaystyle \| \atop \displaystyle H(Du(x,t))} \leq 0$$

so $$u_t(x,t) + H(Du(x,t)) \leq 0 .$$

Supersolution: Choose z s.t.

$$u(x,t) = tL\left(\frac{x-z}{t}\right) + g(z)$$

Fix $h>0$ and set $s=t-h$, $y=\frac{s}{t}x+\left(1-\frac{s}{t}\right)z$

then $\frac{x-z}{t} = \frac{y-z}{s}$ and

$$u(x,t) - u(y,s) \geq tL\left(\frac{x-z}{t}\right) + g(z) - \left[sL\left(\frac{y-z}{s}\right) + g(z)\right]$$

$$u(x,t) - u(y,s) \geq (t-s) L\left(\frac{x-z}{t}\right) \qquad \text{or}$$

$$\frac{u(x,t) - u(x - \frac{h}{t}(x-z), t-h)}{h} \geq L\left(\frac{x-z}{t}\right)$$

letting $h \to 0$

that $\frac{x-z}{t} \cdot Du(x,t) + u_t(x,t) \geq L\left(\frac{x-z}{t}\right)$

and therefore

$$\begin{aligned} u_t(x,t) + H(Du(x,t)) &= u_t(x,t) + \max_{v \in \mathbb{R}^n} \{ v \cdot Du(x,t) - L(v) \} \\ &\geq u_t(x,t) + \frac{x-z}{t} \cdot Du(x,t) - L\left(\frac{x-z}{t}\right) \\ &\geq 0 \end{aligned}$$

$\blacksquare$

<u>Thm</u> The function u given by Hopf-Lax is Lipschitz, differentiable a.e. in $\mathbb{R}^n \times (0,\infty)$ and solves the IVP

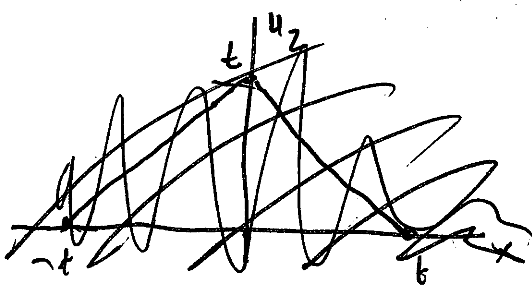
$$\begin{cases} u_t + H(Du) = 0 & \text{a.e. in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

Ex (Non-uniqueness of a.e. solutions)

$$\begin{cases} u_t + \frac{1}{2}|u_x|^2 = 0 & \text{in } \mathbb{R}\times(0,\infty) \\ u = 0 & \text{on } \mathbb{R}\times\{t=0\} \end{cases}$$

$u_1(x,t) = 0$ solves

also $u_2(x,t) = \begin{cases} \overset{x-t}{\cancel{x-t}}\ \cancel{t-x} & \cancel{x-}\ 0 \le x \le t \\ \cancel{x+t}\ -x-t & -t \le x \le 0 \\ 0 & |x| \ge t. \end{cases}$



only 0 solution a.e.

~~doesn't solve~~ not diff'able on $x = 0, \pm t$.

or anything like

ϕ w/ $|\phi'| = 1$ a.e. and $\phi \ge 0$

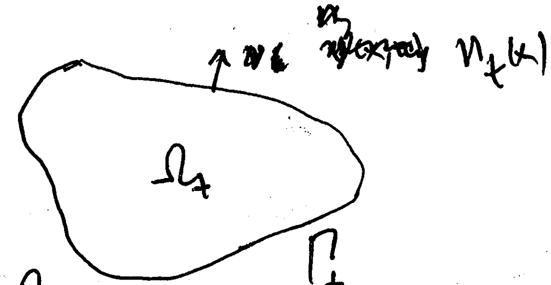
$u(x,t) = (\phi(x) - t)_-$ satisfies $u(x,0) = 0$

and $\partial_t u + H(Du) = 0$ a.e. in $\mathbb{R}^n \times (0,\infty)$.

Some Examples of H-J eqns

Level-set evolutions:

~~Idea~~ We are interested in making a formulation of



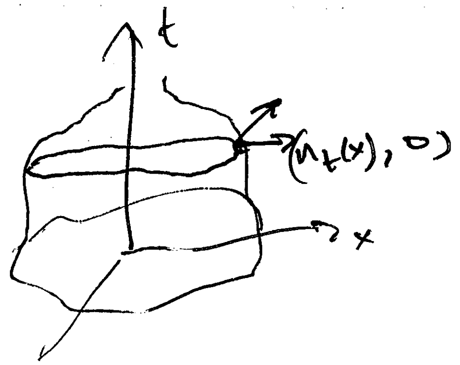
hypersurface Γ_t = bdry of domain Ω_t

outer normal $n_t : \Gamma_t \to \partial B(0,1)$.

Want to solve the PDE, Γ_t moves w/ normal velocity

$$\boxed{V_n = c(x,t)}$$

How to interpret this PDE?



$$\Omega_{0,T} = \bigcup_{0 \le t \le T} \Omega_t$$

w/ $\mathbb{R}^n \times \mathbb{R}$ outer normal

$$\nu(x,t) = \frac{(n_t(x), m(x,t))}{\sqrt{1 + m(x,t)^2}}$$

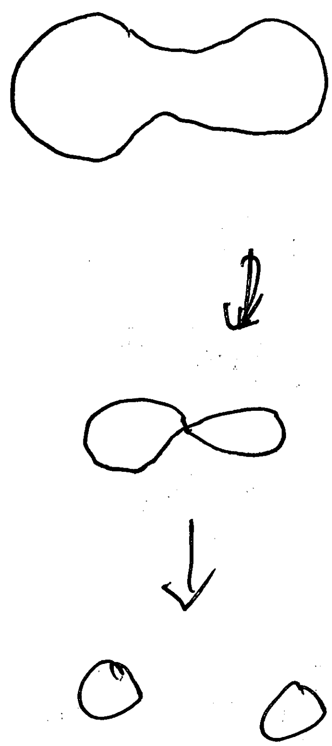
$-m$ is the outward normal velocity of Γ_t

another interpretation:

X_t any path lying in Γ_t through $X_{t_0} = x_0$

then $\dot{X}_{t_0} \cdot n_{t_0}(x_0)$ is the outward normal velocity of Γ_t at x_0.

The problem w/ this sort of solution

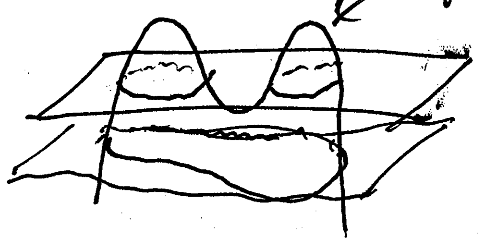
$t=0$  Γ_0, $V_n = -1$

topological changes ~~which~~ are difficult to deal with.

The idea is to view Γ_t as the zero level set of a fcn $u(x,t) : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$.

$$\Gamma_t = \{ x \in \mathbb{R}^n : u(x,t) = 0 \},$$

topological changes are easier



how is outward normal velocity encoded in u?

X_t path in Γ w/ $X_t = x$

$$0 = \frac{d}{dt} u(X_t, t) = \cancel{u_t + \dot{X}_t \cdot Du(X_t,t)}$$

$$= u_t(X_t, t) + \dot{X}_t \cdot Du(X_t, t)$$

$$= u_t(X_t, t) + \dot{X}_t \cdot \frac{Du}{|Du|} |Du| (X_t, t)$$

$$= u_t(X_t, t) + V_n |Du| (X_t, t)$$

(as long as $u < 0$ in Ω_t, $u > 0$ in $\mathbb{R}^n \setminus \Omega_t$, $\left.\frac{Du}{|Du|}\right|_{(X,t)} = +n_t(x)$)

so $$\boxed{\frac{u_t}{|Du|}(x,t) = -V_n}$$

so $$V_n = c(x,t) \longrightarrow -\frac{u_t}{|Du|} = c(x,t)$$

<u>or</u>

the H-J eqn $$u_t + c(x,t)|Du| = 0.$$

example $\quad u_t + |Du| = 0 \quad$ w/ $\quad u(x,0) = g(x)$

Hopf-Lax (formal) solution

$$L(v) = H^*(v) = \sup_{p\in\mathbb{R}^n}\{v\cdot p - H(p)\} = \sup_{p\in\mathbb{R}^n}\{v\cdot p - |p|\}$$

$$= \begin{cases} 0 & |v|\le 1 \\ +\infty & |v| > 1 \end{cases}$$

so the associated variational problem is

$$u(x,t) = \inf\left\{ g(y) : \begin{array}{l} w \text{ smooth path w/} \\ |\dot w|\le 1 \text{ on } [0,t] \\ w(0)=y,\ w(t)=x \end{array}\right\}$$

Hopf-Lax

$$u(x,t) = \inf_{y\in\mathbb{R}^n}\left\{tL\left(\frac{x-y}{t}\right) + g(y)\right\} = \inf_{|y-x|\le t} g(y)$$

Suppose $\quad g(x) = \cancel{\text{dist}}\ \text{sign dist}(x, \Gamma_0) = \begin{cases} \text{dist}(x,\Omega_0) & x\in\Omega_0^c \\ -\text{dist}(x,\Omega_0^c) & x\in\Omega_0 \end{cases}$

then $\quad u(x,t) = \inf_{|y-x|\le t} \text{sign dist}(x,\Gamma_0)$

Claim: $\quad u(x,t) = \left(\text{sign dist}(x,\Gamma_0) - t\right) \vee \left(\min_{y\in\mathbb{R}^n} g(y)\right)$

proof of Claim

clearly $u(x,t) = \inf_{|y-x|\le t} g(y) \geq \inf_{\mathbb{R}^n} g$

and if $x_* = \arg\min_{\mathbb{R}^n} g$ satisfies $|x_*-x|\le t$

then $u(x,t) = \min_{\mathbb{R}^n} g$

Call ~~otherwise~~ $E = \{x\in\mathbb{R}^n : g(x) = \min_{\mathbb{R}^n} g\}$

Note that $\operatorname{sign\,dist}(x,\Gamma_0) = \operatorname{dist}(x,E) + \cancel{\operatorname{dist}(E,\Gamma_0)}\ \min_{\mathbb{R}^n} g$

$$u(x,t) = \min_{|y-x|\le t} \operatorname{signdist}(x,\Gamma_0) = \min_{|y-x|\le t} \operatorname{dist}(x,E) + \min_{\mathbb{R}^n} g$$
$$= \Big(\operatorname{dist}(x,E) - t\Big) \vee 0 + \min g$$

Viscosity Solutions of H-J Eqns

~~Def~~

We need to make some definitions first

$D_+u(x) = \{ p \in \mathbb{R}^n : u(y) \le u(x) + p\cdot(y-x) + o(|y-x|)$ as $y \to x \}$

Superdifferential

$D_-u(x) = \{ p \in \mathbb{R}^n : u(y) \ge u(x) + p\cdot(y-x) + o(|y-x|)$ as $y \to x \}$

subdifferential

From the proof of Hopf-Lax solution solves

$$u_t + H(Du) = 0$$

where u is differentiable,

we actually could have shown

$(p,\tau) \in D_+^{x,t} u(x,t) \Rightarrow \tau + H(p) \le 0$ "subsolution"

$(p,\tau) \in D_-^{x,t} u(x,t) \Rightarrow \tau + H(p) \ge 0$ "supersolution"

Lem (properties) (i) u is diff'able at x,

$\iff D_+u(x) = D_-u(x) = \{Du(x)\}$

(ii) $D_+u(x)$, $D_-u(x)$ both convex sets

(iii) if $D_+u(x)$ non-empty

$\Rightarrow$ either $D_+u(x) = D_-u(x) = \{Du(x)\}$

or $D_-u(x) = \phi$

~~We say~~

Statements are a bit easier for non-time dependent

$$(H\text{-}J) \begin{cases} H(Du) = 0 & \text{in } U \\ \text{bdry data} \end{cases}$$

Def u is a viscosity soln of $(H\text{-}J)$ if

u is continuous and

(i) subsolution, $\forall\ p \in D_+u(x)$

$$H(p) \leq 0$$

(ii) supersolution, $\forall\ p \in D_-u(x)$

$$H(p) \geq 0 .$$

Rem 1: This is in a sense the L^∞ version of the weak solution condition we saw before.

Rem 2:

Rmk 2: Although it is not immediately obvious from this definition, the definition of viscosity solution essentially says that u satisfies a "local comparison principle" with respect to C^1 test functions.

Def 2 u is a viscosity ~~subsoln~~ soln of (HJ) ~~if~~ if

for every ~~$\phi \in C^1(\mathbb{R}^n \times [0,\infty))$~~ $(x,t) \in \mathbb{R}^n \times (0,\infty)$

every parabolic cylinder $Q_r(x,t) = \{(y,s) : |y-x| \le r, \; t-r < s \le t\}$

and any $\phi \in C^1(Q_r(x,t))$

(i) if ϕ touches u from below at (x,t) in $Q_r(x,t)$ then

$$\phi_t + H(D\phi) \geq 0$$

(ii) if ϕ touches u from above at (x,t) in $Q_r(x,t)$ then

$$\phi_t + H(D\phi) \leq 0 .$$

example: $\begin{cases} |Du| = 1 & \text{in } (0,1) \\ u = 0 & \text{at } \{0,1\} \end{cases}$

$u(x) = \frac{1}{2} - |x - \frac{1}{2}|$ satisfies

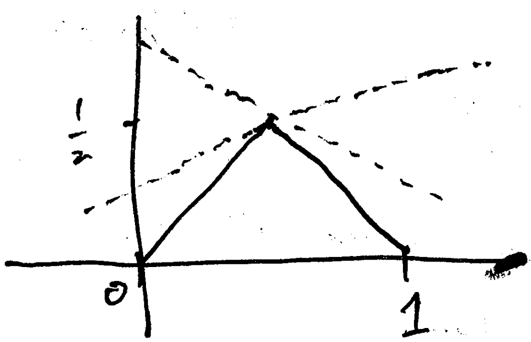
u diff'able at $x \in (0,1) \setminus \{\frac{1}{2}\}$

$|Du(x)| = 1$ ok ✓

at $x = \frac{1}{2}$

$D_+ u(\frac{1}{2}) = [-1,1]$, $D_- u(\frac{1}{2}) = \phi$

$\forall$ $p \in [-1,1]$ $|p| - 1 \leq 0$ ✓ subsoln condition



Thm: if u, viscosity subsolution, v viscosity supersoln of (HJ) in $\mathbb{R}^n \times (0,\infty)$ w/

$u(x,0) \leq v(x,0)$ then

$u(x,t) \leq v(x,t)$ $\forall$ $t > 0$.