Hamilton - Jacobi Equations

$$(1) \quad \begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) & x \in \mathbb{R}^n \end{cases}$$

$H : \mathbb{R}^n \to \mathbb{R}$ called the Hamiltonian

our goal as w/ the conservation laws is to continue the solution past the time when characteristics cross. To do so it will be ~~the~~ necessary, as before, to select the correct weak solution ~~correctly~~ by using some physical principle.

We will find that (1) PDE is associated w/ an optimal control problem

Characteristics for (1) are

$$\begin{cases} \dot{x} = D_p H(p,x) \\ \dot{p} = -D_x H(p,x) \end{cases}$$ called Hamilton's ODEs

these ODE arise in classical mechanics from a variational principle

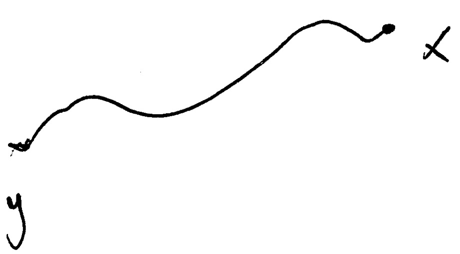
let $L : \mathbb{R}^n \to \mathbb{R}$ a given smooth fn
called a Lagrangian we write $L(q, x)$.
or Running Cost

fix $x, y \in \mathbb{R}^n$ and $t > 0$ the action
or cost functional is defined

$$I[w] = \int_0^t L(\dot{w}(s), w(s))\, ds$$ defined for

$$w \in \mathcal{A} = \{ w(\cdot) \in C^2([0,t]; \mathbb{R}^n) : w(0) = y,\ w(t) = x \}$$

smooth paths from y to x



we wish to find the minimal cost/action path from y to x.

$$I[x(\cdot)] = \min_{w \in \mathcal{A}} I[w(\cdot)].$$

let us assume we can find such a path,

$x(\cdot)$

then ~~Φ~~

Thm (Euler-Lagrange Eqns) The fcn x solves

$$\boxed{-\frac{d}{ds}\left(D_q L(\dot{x},x)\right) + D_x L(\dot{x},x) = 0 \qquad 0 \le s \le t.}$$

proof: as we did for previous variational principles we look at the "directional derivative" of I in direction $\phi \in \tan_x A$.

let v smooth $[0,t] \to \mathbb{R}^n$,

$\phi(0) = \phi(t) = 0$ so that

$w = x + \varepsilon\phi \in A$

~~$I[w]$~~ $\quad I[x] \le I[w] \quad \forall\, \varepsilon > 0.$

so $\quad i(\varepsilon) = I[x+\varepsilon\phi]$ has a min at $\varepsilon = 0$

$\Rightarrow \quad i'(0) = 0$ if it exists.

$$i(\varepsilon) = \int_0^t L(\dot{x}+\varepsilon\dot{\phi}, x+\varepsilon\phi, \cancel{s})\, ds \quad \text{so}$$

$$i'(\varepsilon) = \int_0^t D_q L(\dot{x}+\varepsilon\dot{\phi}, x+\varepsilon\phi)\dot{\phi} + D_x L(\dot{x}+\varepsilon\dot{\phi}, x+\varepsilon\phi)\phi\, ds$$

setting $\varepsilon = 0$,

$$0 = i'(0) = \int_0^t D_q L(\dot{x}, x)\dot{\phi} + D_x L(\dot{x}, x)\phi \, ds$$

and integrate by parts using

$$\phi(0) = \phi(t) = 0$$

$$= \int_0^t \left(-\frac{d}{ds}(D_q L(\dot{x}, x)) + D_x L(\dot{x}, x) \right) \phi \, ds$$

since this is zero for all $\phi \in C_c^2((0,t); \mathbb{R}^n)$

we obtain

$$\begin{cases} -\frac{d}{ds}(D_q L(\dot{x}, x)) + D_x L(\dot{x}, x) = 0 \\ \text{for } s \in (0,t) \end{cases}$$

$\square$

Example 1: $L(q,x) = \frac{1}{2}m|q|^2 - \phi(x)$

kinetic energy potential energy

E-L eqn is Newton's law

$$\boxed{m\ddot{x} = -D\phi(x)}$$

Example 2: $L(q,x) = \begin{cases} 0 & |q| \le 1 \\ +\infty & |q| > 1 \end{cases}$

cost functional is minimal time to move from y to x w/ speed ≤ 1. E-L eqn is harder to interpret.

A payoff problem

define

the running cost

$$u(x,t) = \inf\left\{ \int_0^t L(\dot{w}(s))\,ds + g(y) \;\middle|\; w(0)=y,\ w(t)=x \right\}$$

the value function

the payoff / terminal cost

optimal trajectories solve the Euler-Lagrange Eqn

$$\frac{d}{ds}\left(D_v L(\dot{x})\right) = 0 \qquad 0 \le s < t.$$

~~dynamic programming principle~~, $\forall\ 0 < s \le t$,

$$u(x,t) = \inf\left\{ \int_s^t L(\dot{w}(\tau), w(\tau))\,d\tau + u(y,s) \;\middle|\; w(s)=y,\ w(t)=x \right\}$$

using the DPP infinitesimally

$$u(x,t+h) = \inf_w \left\{ \int_t^{t+h} L(\dot{w}(\tau), w(\tau))\,d\tau + u(y,t) \;\middle|\; w(t)=y,\ w(t+h)=x \right\}$$

$$= \inf_v \left\{ hL(v,x) + u(x-hv,t) \right\} + O(h^2)$$

$$= \inf_v \left\{ hL(v,x) + u(x,t) - h\,v\cdot Du(x,t) \right\} + O(h^2)$$

$$= u(x,t) + h \inf_v \left\{ -v\cdot Du(x,t) + L(v,x) \right\} + O(h^2)$$

on

$$\partial_t u = \inf_v \{ -v \cdot Du(x,t) + L(v,x) \}$$

or

$$\partial_t u + \underbrace{\sup_v \{ v \cdot Du(x,t) - L(v,x) \}}_{=: H(Du,x)} = 0$$

<u>proof of DPP</u> :

$$u(x,t) = \inf \left\{ \int_0^t L(\dot{w}, w)\, d\tau + g(y) \mid w(0)=y,\ w(t)=x \right\}$$

let $y \in \mathbb{R}^n$,

(9) take w_* optimal for $u(y,s)$, i.e.

$$u(y,s) = \int_0^s L(\dot{w}_*, w_*)\, d\tau + g(w_*(0))$$

and let
$$\bar{w}(\tau) = \begin{cases} w(\tau) & s \le \tau \le t \\ w_*(\tau) & 0 \le \tau \le s \end{cases}$$

with w any path s.t. $w(s)=y$, $w(t)=x$

then
$$u(x,t) \le \int_0^t L(\dot{\bar{w}}, \bar{w})\, d\tau + g(\bar{w}(0))$$

$$= u(y,s) + \int_s^t L(\dot{w}(\tau), w(\tau))\, d\tau$$

taking inf over w yields

$$u(x,t) \le \inf \left\{ \int_s^t L(\dot{w}(\tau), w(\tau))\, d\tau + u(y,s) \mid w(s)=y,\ w(t)=x \right\}$$

for the other direction let w_* optimal

for $$u(x,t) = \int_0^t L(\dot{w}_*, w_*)\,d\tau + g(w_*(0))$$

$$= \int_s^t L(\dot{w}_*, w_*)\,d\tau + \int_0^s L(\dot{w}_*, w_*)\,d\tau + g(w_*(0))$$

$$\geq \int_s^t L(\dot{w}_*, w_*)\,d\tau + u(w_*(s), s)$$

$$\Rightarrow \quad u(x,t) \geq \inf\left\{ \int_s^t L(\dot{w}, w)\,d\tau + u(y,s) : w(s) = y,\ w(t) = x \right\}$$

$\square$

Hamilton's ODE's

Suppose that x is an action minimizer (or critical point) call

$$p(s) = D_q L(\dot{x}(s), x(s)) \quad 0 \le s \le t.$$

$p(\cdot)$ is called the generalized momentum

x - position, $\dot{x}$ - velocity.

Now we suppose that $\forall\ x, p \in \mathbb{R}^n$

$$\left\{ \begin{array}{l} p = D_v L(v, x) \text{ can be uniquely solved} \\ \text{for } v \text{ as a smooth fcn of } p \text{ - } x \\ \qquad v = v(p, x). \end{array} \right.$$

Def The Hamiltonian associated with the Lagrangian L is

$$H(p, x) = p \cdot v(p, x) - L(v(p, x), x)$$

Legendre Transform

Now we put some additional assumptions

(1) $v \mapsto L(v)$ is convex

(2) $\lim_{|q| \to \infty} \frac{L(v)}{|v|} = +\infty$ coercivity

Def The Legendre transform of L is

$$L^*(p) = \sup_{q \in \mathbb{R}^n} \{ p \cdot q - L(q) \} \qquad (p \in \mathbb{R}^n)$$

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suppose the supremum is achieved by $q_* \in \mathbb{R}^n$

$$L^*(p) = p \cdot q_* - L(q_*)$$ and

$q \mapsto p \cdot q - L(q)$ has maximum at q_*

then $p = DL(q)$ is solvable for q

(q_* solves) although not necessarily

uniquely $L^*(p) = \cancel{p \cdot q(p) - L(q(p))}$

$$p \cdot q(p) - L(q(p))$$

which is the Hamiltonian H associated w/ L.

We thus call $H = L^*$.

Now conversely given Hamiltonian H can we find L?

<u>Thm</u> (Convex duality) Assume L convex, coercive and define $H = L^*$ then

(1) H is convex and coercive

(2) $H^* = L$

Rem: We call H and L <u>convex dual functions</u>

<u>proof</u>:

· $p \mapsto p \cdot q - L(q)$ is linear

so $$H(p) = \sup_{q \in \mathbb{R}^n} \{ p \cdot q - L(q) \}$$ is convex.

· $$H(p) = \sup_{q \in \mathbb{R}^n} \{ q \cdot p - L(q) \}$$

take $q = \frac{Rp}{|p|}$ $$\geq R|p| - L\left(\frac{Rp}{|p|}\right)$$

$$\geq R|p| - \max_{B(0,R)} L \qquad \text{(coercive)}$$

$$\Rightarrow \lim_{|p| \to \infty} \frac{H(p)}{|p|} \geq R \qquad \text{for every } R > 0$$

(duality):

$\forall$ p, q

$$H(p) + L(q) \geq p \cdot q$$

so $$L(q) \geq \sup_{p \in \mathbb{R}^n} \{ p \cdot q - H(p) \} = H^*(q)$$

and

$$H^*(q) = \sup_{p \in \mathbb{R}^n} \{ p \cdot q - \sup_{r \in \mathbb{R}^n} \{ r \cdot p - L(r) \} \}$$

$$= \sup_{p \in \mathbb{R}^n} \inf_{r \in \mathbb{R}^n} \{ p \cdot (q - r) + L(r) \}$$

Since L convex $\Rightarrow$ it has a supporting hyperplane at q, $\exists$ $s \in \mathbb{R}^n$ s.t.

$$L(r) \geq L(q) + s \cdot (r - q) \quad \forall \; r \in \mathbb{R}^n$$

taking $p = s$ in the supremum

$$H^*(q) \geq \inf_{r \in \mathbb{R}^n} \{ s \cdot (q - r) + L(r) \}$$

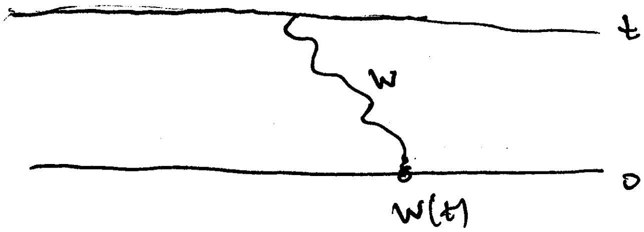
$$\geq L(q) \qquad \text{from choice of } s.$$

$\mathbb{R}$

The Hopf-Lax formula

define

(1) $$u(x,t) = \inf\left\{ \int_0^t L(\dot{w}(s))\,ds + g(y) \;\middle|\; w(0)=y,\ w(t)=x \right\}$$



will turn out to solve (in a sense)

$$\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) \end{cases}$$

where $H = L^*$ or $L = H^*$ depending on our starting point.

<u>Thm</u> (Hopf-Lax formula)

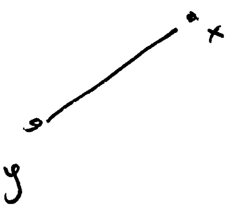
Let $x \in \mathbb{R}^n$, $t > 0$ then u as in (1) is

$$\boxed{u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ t L\left(\frac{x-y}{t}\right) + g(y) \right\}}$$

(i.e. optimal trajectories in (1) are straight lines) proof of Hopf-Lax formula :

one direction is easy

take $w(s) = y + \frac{s}{t}(x-y) \qquad 0 \le s \le t$



$$u(x,t) \le \int_0^t L(\dot w(s))\,ds + g(y)$$
$$= \int_0^t L\left(\frac{x-y}{t}\right) ds + g(y)$$
$$= t L\left(\frac{x-y}{t}\right) + g(y)$$

take inf over $y \in \mathbb{R}^n$ to obtain

$$u(x,t) \le \inf_{y \in \mathbb{R}^n} \left\{ t L\left(\frac{x-y}{t}\right) + g(y) \right\}$$

if w is any C^1 fcn $w(t) = x$ then

calling $y = w(0)$

$$L\left(\frac{1}{t}\int_0^t L(\dot w(s))\,ds\right) \le \frac{1}{t}\int_0^t L(\dot w(s))\,ds \qquad \text{(Jensen)}$$

so

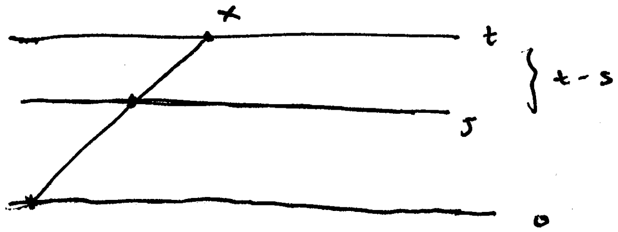
$$u(x,t) \ge \inf_{y \in \mathbb{R}^n} \left\{ t L\left(\frac{1}{t}\int_0^t \dot w(s)\,ds\right) + g(y) \right\}$$
$$= \inf_{y \in \mathbb{R}^n} \left\{ t L\left(\frac{x-y}{t}\right) + g(y) \right\} \qquad \square$$

(use) super-linear growth of L to show inf is achieved.

Lem (Dynamic Programming Principle)

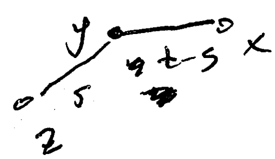
For each $x \in \mathbb{R}^n$, $0 \le s < t$ we have

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ (t-s) L\left(\frac{x-y}{t-s}\right) + u(y,s) \right\}$$



proof: 1. Fix $y \in \mathbb{R}^n$, $0 < s < t$ and choose z so that

$$u(y,s) = sL\left(\frac{y-z}{s}\right) + g(z)$$



Since L convex, $\frac{x-z}{t} = \left(1-\frac{s}{t}\right)\frac{x-y}{t-s} + \frac{s}{t}\frac{y-z}{s}$

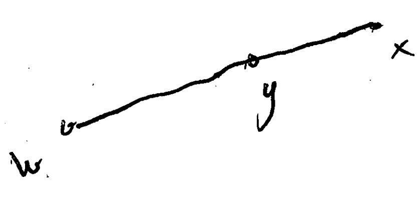
$$L\left(\frac{x-z}{t}\right) \le \left(1-\frac{s}{t}\right) L\left(\frac{x-y}{t-s}\right) + \frac{s}{t} L\left(\frac{y-z}{s}\right) \quad \text{so}$$

$$u(x,t) \le tL\left(\frac{x-z}{t}\right) + g(z) \le (t-s)L\left(\frac{x-y}{t-s}\right) + sL\left(\frac{y-z}{s}\right) + g(z)$$

$$= (t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s)$$

for each $y \in \mathbb{R}^n$.

Now choose w s.t. $u(x,t) = tL\left(\frac{x-w}{t}\right) + g(w)$



set $y = \frac{s}{t}x + \left(1-\frac{s}{t}\right)w$