Traveling wave solutions of PDE

(Chapter 12 of Shearer & Levy)

heuristic description:

look for PDE solutions of the form

$u(x,t) = f(x-ct)$, $\xi = x - ct$

PDE $\rightarrow$ ODE for f

This ODE will have equilibria (corresponding to constant solns of PDE)

travelling waves correspond to connections between equilibria. (heteroclinic orbits or homoclinic (?))

Burger's Eqn (viscous)

$u_t + u u_x = \varepsilon u_{xx}$ (Burgers)

$\varepsilon > 0 \rightsquigarrow$ viscosity so $\varepsilon u_{xx} \rightsquigarrow$ dissipation

$u u_x \rightsquigarrow$ self-transport (nonlinear)

look for $u(x,t) = f\left(\frac{x-ct}{\varepsilon}\right)$ $\xi = \frac{x-ct}{\varepsilon}$

conditions at $\pm\infty$

$$f(\pm\infty) = u_\pm \quad , \quad f'(\pm\infty) = 0$$

substituting in $u(x,t) = f\left(\frac{x-ct}{\varepsilon}\right)$ we get

$$-\frac{c}{\varepsilon} f' + \frac{1}{\varepsilon} f f' = \varepsilon \cdot \frac{1}{\varepsilon^2} f''$$

or $$-cf' + \left(\frac{f^2}{2}\right)' = f''$$

$$\cancel{-cf' + ff' = f''}$$

which can be integrated,

$\int_{-\infty}^{\xi}$ (equation) to obtain

$$-c(f - u_-) + \frac{1}{2}\left(f^2 - u_-^2\right) = f'$$

since $f'(+\infty) = 0$

$$\Downarrow$$

$$-c(u_+ - u_-) + \frac{1}{2}(u_+^2 - u_-^2)$$

$\Rightarrow$ either $u_+ = u_-$ <u>or</u>

$$\boxed{c = \frac{\frac{1}{2}u_+^2 - \frac{1}{2}u_-^2}{u_+ - u_-}}$$

Note: this is exactly the Rankine-Hugoniot jump condition from the non-viscous Burger's Eqn.

if $u_+ = u_-$ then $f(\xi) \equiv u_-$

(trivial solution $\Rightarrow$ no solitons)

if $u_+ \neq u_-$ more interesting, need

$c = \frac{1}{2}(u_+ + u_-)$ and then ~~there~~

~~is a non trivial solution of~~

~~$f' = (f - u_-)\left(\frac{1}{2}(f + u_+) - c\right)$~~

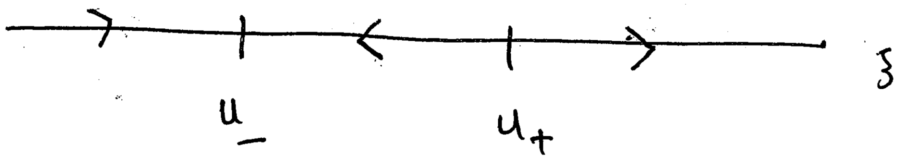
~~note solution independent of ε~~

~~so $f\left(\frac{x - ct}{\varepsilon}\right) \xrightarrow[\varepsilon \to 0]{}$ solution of Riemann problem for Burger's eqn.~~

so the ODE becomes

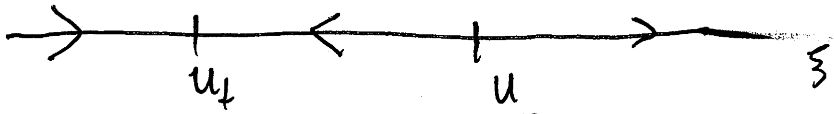
$$f' = \frac{1}{2}(f - u_-)(f - u_+)$$ which can be easily understood by phase line analysis

if $u_- < u_+$



$\Rightarrow$ no solution exists w/ $f(-\infty) = u_-$, $f(+\infty) = u_+$

if $u_+ < u_-$



$\Rightarrow$ $\exists$ solution w/

$f(-\infty) = u_-$, $f(+\infty) = u_+$

could even solve using partial fractions I think.

Note: when $\boxed{c = \frac{1}{2}(u_+ + u_-)}$ ~~(Rankine - Hugoniot)~~

and

$$\boxed{u_+ < c = \frac{u_+ + u_-}{2} < u_-}$$

(Entropy condition)

$\Rightarrow$ $u^\varepsilon(x,t) = f\left(\frac{x-ct}{\varepsilon}\right)$ travelling wave s-ln

of $(\text{Burgers})_\varepsilon$ $\forall$ $\varepsilon > 0$

as $\varepsilon \to 0$ $u^\varepsilon(x,t) \to$ Entropy satisfying shock solution of Burgers.

The Korteweg-de Vries Eqn

The KdV eqn

$$(\text{KdV}) \quad \left[\, u_t + uu_x + \gamma^2 u_{xxx} = 0 \right.$$

Conserves: (actually there are ∞ly many integ. invariants)

Momentum $= \int_{-\infty}^{\infty} u \, dx$

Kinetic Energy $= \int_{-\infty}^{\infty} u^2 \, dx$

famous for solitary travelling waves called solitons

which can pass through each other

unchanged even though the eqn is nonlinear.

Dispersion relation

The KdV eqn is an example of a dispersive eqn, higher frequency waves travel with a higher velocity.

let's linearize by looking a a solution

$$u(x,t) = \cancel{\varepsilon v(x,t)} \quad 1 + \varepsilon v(x,t)$$

$$\varepsilon v_t + (1+\varepsilon v)\varepsilon v_x + \gamma \varepsilon v_{xx} = 0$$

$$\varepsilon (v_t + v_x + \gamma v_{xxx}) + \cancel{\varepsilon^2 v v_x} = 0$$

ignore when ε small

$$v_t + v_x + \gamma v_{xxx} = 0$$

linearized KdV

look for solutions $\boxed{v(x,t) = u(t) e^{i\xi x}}$

ξ - wave number (~ frequency)

wavelength / period ~~frequency~~ $= \dfrac{2\pi}{\xi}$

substituting in we get

$$\varphi' + i\xi\varphi + \gamma(i\xi)^3\varphi = 0$$

or

$$\varphi' = -i(\xi - \gamma\xi^3)\varphi$$

so

$$\varphi(t) = e^{-i(\xi-\gamma\xi^3)t} \qquad \cancel{\varphi(0)} = 1$$

so we find a family of solutions

$$v(t,x) = e^{\lambda t} e^{i\xi x} \quad \text{w/} \quad \underbrace{\boxed{\lambda(\xi) = -i(\xi - \gamma\xi^3)}}_{\text{dispersion relation}}$$

$$= e^{i\xi(x - (1-\gamma\xi^2)t)}$$

travelling wave w/ speed

$$\boxed{c(\xi) = 1 - \gamma\xi^2}$$ depending on the wave <u>number</u>

this is <u>dispersion</u>

(note $c(0) = 1$ since we linearized about $u(t,x) = 1$ stationary solution)

Solitons of KdV

look for $u(x,t) = f(x-ct)$

suppose $f(\pm\infty) = 0$

plug into the PDE to get

$$-cf' + ff' + \gamma f''' = 0$$

and integrate (assuming $f^{(k)}(\pm\infty) = 0$) ~~for $k \geq 2$~~

to find

$$\boxed{-cf + \frac{1}{2}f^2 + \gamma f'' = 0}$$

This is a Hamiltonian system ~~if we~~

w/ $\cancel{H(v', v) = \frac{\gamma}{2}(f')^2 + f}$

$$\boxed{H(f', f) = \frac{1}{2}(f')^2 + \frac{1}{6\gamma}f^3 - \frac{c}{2\gamma}f^2}$$

We can see that $H(f', f)$ is conserved by multiplying by f' and integrating the ODE.

Call $p = f'$, $q = f$

then

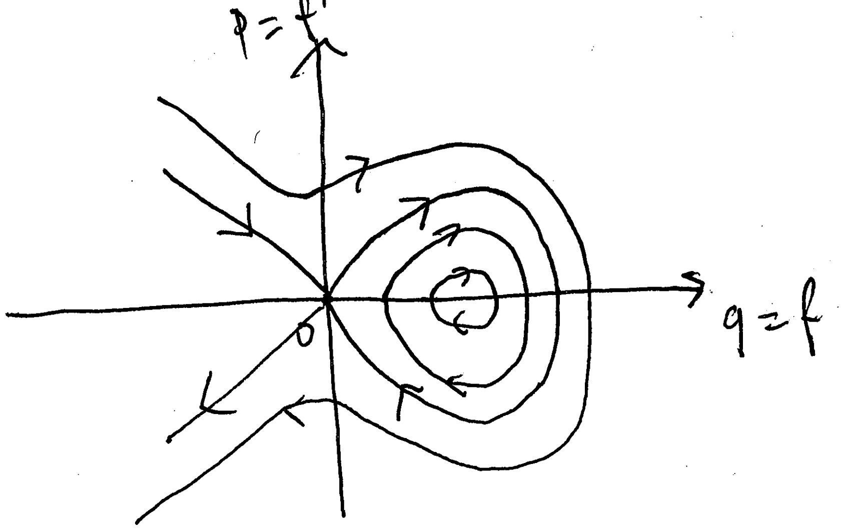
$$\begin{cases} q' = H_p \\ p' = -H_q \end{cases}$$ and (as before)

$H(p(\xi), q(\xi))$ is constant on trajectories

$$\frac{d}{d\xi} H(p,q) = H_p p' + H_q q'$$

$$= -H_q H_p + H_q H_p = 0$$

for hamiltonian systems it is easy to draw the phase plane, just draw level set diagram of H.



$$H(p,q) = \frac{1}{2}p^2 + \frac{1}{6\gamma}q^3 - \frac{c}{2\gamma}q^2$$

~~$H \to +\infty$~~

~~$p, q \to \infty$~~

~~but $H < 0$~~

~~$H \to$~~

the trajectory joining 0 to itself is called a homoclinic orbit

corresponds to a soliton solution of KdV by $\xi = x - ct$.

~~aside:~~ ~~along the desired orbit~~

$$\cancel{0 = H(f', f) = \frac{1}{2}(f')^2 + \frac{1}{6\gamma}f^3 - \frac{1}{2\gamma}f^2}$$

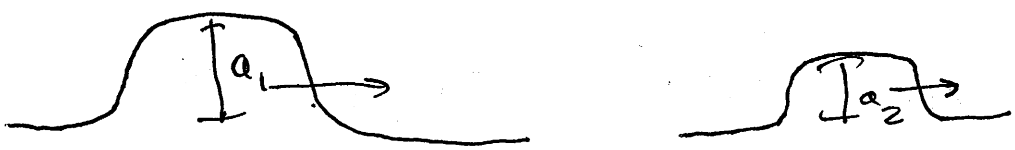
$$\cancel{(f')^2 = \frac{1}{\gamma}f^2\left(c - \frac{1}{3}f\right)}$$

$$\cancel{f' = \pm\frac{1}{\gamma}f\left(c - \frac{1}{3}f\right)^{1/2}}$$

<u>fyi</u> soliton is

$$\boxed{u(x,t) = a \,\mathrm{sech}^2\left(\sqrt{\frac{a}{12\gamma}}\,(x-ct)\right) \quad \text{w/} \quad c = \frac{a}{3}}$$

note that higher amplitude waves travel faster.

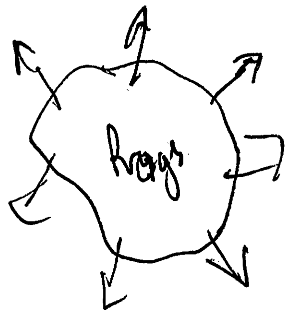


nonlinear interaction

solitons pass through "unchanged"

Fisher-KPP Equation

population growth and diffusion

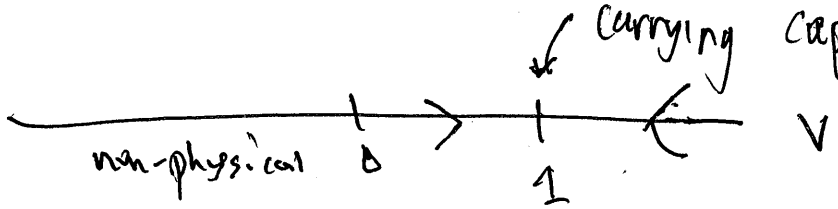


low density of frogs

population grows faster in areas with low density → leads to spatial propagation of the population faster than just diffusion would result in

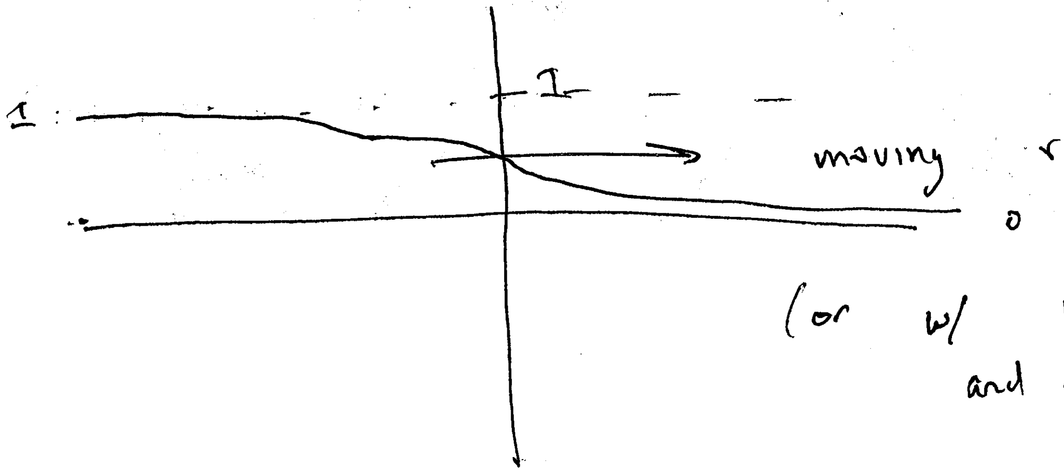
$$\underbrace{u_t - \Delta u}_{\text{spatial diffusion}} = \underbrace{u(1-u)}_{\text{logistic growth model}} \qquad \text{Fisher-KPP}$$

phase line for $\dot{v} = v(1-v)$



note: (we have assumed carrying capacity independent of x in this model) expect a travelling wave connecting the stationary solutions

$u(x,t) = 1$ and $u(x,t) = 0$



(or w/ 1, 0 reversed and speed reversed).

Substitute in $u(x,t) = f(x-ct)$

$$-cf' = f'' + f(1-f)$$

~~then~~ or as a 1st order system

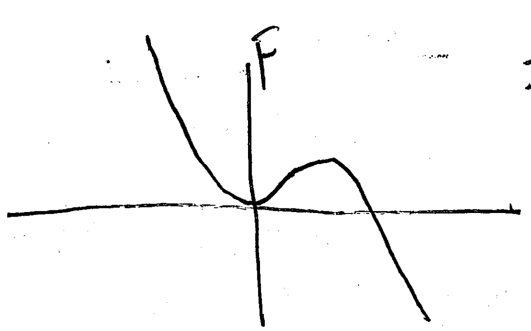
w/ $u = f$
$v = f'$

$$\begin{cases} u' = v \\ v' = -cv - u(1-u) \end{cases}$$

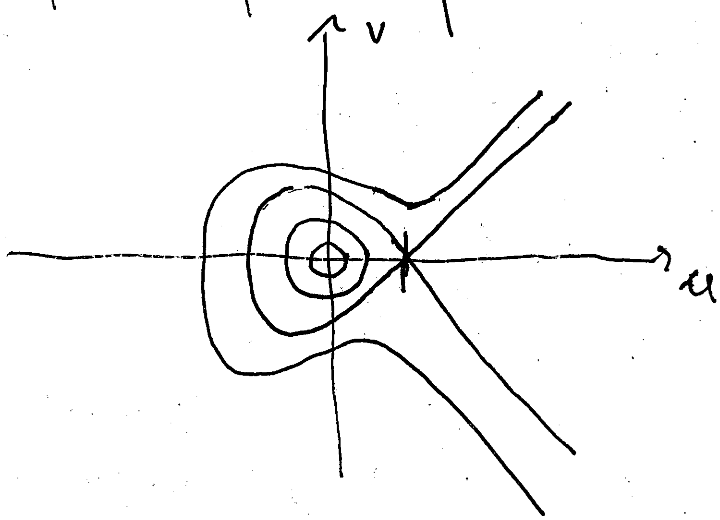
when $c=0$ this is Hamiltonian system

with $H(v,u) = \frac{v^2}{2} + \frac{u^2}{2} - \frac{u^3}{3}$

$= \frac{v^2}{2} + F(u)$



phase portrait for $c=0$



saddle pt at $(0,1)$
stable center at $(0,0)$
oscillatory solutions
(which are non-physical since u doesn't stay positive)

for $c>0$ small then solutions will still be oscillatory and non-physical.

for $c>0$ $\frac{d}{d\xi} H(v,u) = -cv^2 \leq 0$ ~~for $c>0$~~

H decreases along trajectories

linearize at $(0,0)$, for small c expect a get a stable spiral, which is not acceptable

looking for a saddle or stable node

linearization at $(0,0)$ ~~if~~ of

$G(u,v) = (v, -cv - u(1-u))$

$$\boxed{DG(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix} \quad \text{w/ eigenvalues} \quad \lambda_\pm = \frac{1}{2}\left(-c \pm \sqrt{c^2-4}\right)}$$

~~turns out there is travelling wave for any $c \geq 2$.~~